35. A Note on the Normal Generation of Ample Line Bundles on Abelian Varieties

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Let k be an algebraically closed field, let A be an abelian variety defined over k and let L be an ample line bundle on A. It is well known that $L^{\otimes n}$ is normally generated if $n\geqq 3$ (see Koizumi [2] or Sekiguchi [5], [6]). But $L^{\otimes 2}$ is not normally generated in general because $L^{\otimes 2}$ is not very ample in general. For the very ampleness of $L^{\otimes 2}$, the following result is obtained (see Ohbuchi [3]).

Theorem A. *$L^{\otimes 2}$ is not very ample if and only if (A, L) is isomorphic to $(A_1\times A_2, \mathcal{O}(\Theta_1\times A_2+A_1\times D_2))$ where A_1 and A_2 are abelian varieties with* $\dim(A_1)>0$ *and Θ_1 is a theta divisor.*

Our purpose is to give a condition for the normal generation of $L^{\otimes 2}$. The result is as follows:

Theorem. *If* $\operatorname{char}(k)\neq 2$ *and L is a symmetric ample line bundle, then $L^{\otimes 2}$ is normally generated if and only if the origine* 0 *of A is not contained in* $\operatorname{Bs}|L\otimes P_\alpha|$ *for any $\alpha\in\hat{A}_2=\{\alpha\in\hat{A}\,;\,2\alpha=0\}$ where $\hat{A}$ is the dual abelian variety of A, P is the Poincaré bundle on $A\times\hat{A}$, $P_\alpha=P_{|A\times\{\alpha\}}$ for $\alpha\in\hat{A}$ and* $\operatorname{Bs}|L\otimes P_\alpha|$ *is the set of all base points of $L\otimes P_\alpha$.*

To prove this theorem, we need three lemmas.

Lemma 1. *If* $\operatorname{char}(k)\neq 2$ *and L is a symmetric ample line bundle, then $\xi^*(p_1^*L\otimes p_2^*L)\simeq p_1^*(L^{\otimes 2})\otimes p_2^*(L^{\otimes 2})$ where $p_i: A\times A\to A$ is the i-th projection $(i=1,2)$ and $\xi: A\times A\to A\times A$ is defined by $\xi(x,y)=(x+y, x-y)$ for all S-valued points x, y where S is a k-scheme.*

Proof. As $\xi^*(p_1^*L\otimes p_2^*L)_{|A\times\{y\}}\simeq T_y^*L\otimes T_{-y}^*L\simeq L^{\otimes 2}$ for any closed point $y\in A$, therefore $\xi^*(p_1^*L\otimes p_2^*L)\otimes(p_1^*(L^{\otimes 2}))^{-1}\simeq p_2^*M$ for some line bundle M on A by See-Saw theorem. Moreover $\xi^*(p_1^*L\otimes p_2^*L)_{|\{0\}\times A}\simeq L\otimes(-1_A)^*L\simeq L^{\otimes 2}$, hence $M\simeq L^{\otimes 2}$.

Lemma 2. *If* $\operatorname{char}(k)\neq 2$ *and L is an ample line bundle, then*

$$\sum_{\alpha\in\hat{A}_2}\Gamma(A, L\otimes P_\alpha)\xrightarrow{2_A^*}\Gamma(A, 2_A^*L)$$

is an isomorphism.

Proof. This is a well known fact (see Mumford [1]).

Lemma 3. *If L is an ample line bundle, then*

$$\Gamma(A, L^{\otimes n})\otimes\Gamma(A, L^{\otimes m})\longrightarrow\Gamma(A, L^{\otimes(n+m)})$$

is surjective if $n\geqq 2$, $m\geqq 3$.

Proof. See Koizumi [2] or Sekiguchi [5], [6].

Proof of Theorem. If the canonical map $\Gamma(A, L^{\otimes 2})\otimes\Gamma(A, L^{\otimes 2})\to\Gamma(A, L^{\otimes 4})$ is surjective, then $L^{\otimes 2}$ is normally generated by Lemma 3. Hence we prove that the canonical map $\Gamma(A, L^{\otimes 2})\otimes\Gamma(A, L^{\otimes 2})\to\Gamma(A, L^{\otimes 4})$ is surjective if and only if the origine 0 of A is not contained in $\mathrm{Bs}\,|L\otimes P_\alpha|$ for any $\alpha\in\hat{A}_2$. Since L is symmetric, there exists an isomorphism $2_A^*L\simeq L^{\otimes 4}$ (see Mumford [1]). As $(L\otimes P_\alpha)^{\otimes 2}\simeq L^{\otimes 2}$ for any $\alpha\in\hat{A}_2$, therefore we obtain the following commutative diagram:

$$\begin{array}{ccc}\Gamma(p_1^*(L\otimes P_\alpha)\otimes p_2^*(L\otimes P_\alpha)) & \xrightarrow{\xi^*} & \Gamma(p_1^*(L^{\otimes 2})\otimes p_2^*(L^{\otimes 2}))\\ {\scriptstyle 2_{A\times A}^*}\downarrow & & \downarrow{\scriptstyle \xi^*}\\ \Gamma(p_1^*(2_A^*L)\otimes p_2^*(2_A^*L)) & \xrightarrow{\simeq} & \Gamma(p_1^*(L^{\otimes 4})\otimes p_2^*(L^{\otimes 4})).\end{array}$$

By Künneth's formula, we obtain the following commutative diagram:

$$\begin{array}{ccc}\Gamma(L\otimes P_\alpha)\otimes\Gamma(L\otimes P_\alpha) & \xrightarrow{\xi^*} & \Gamma(L^{\otimes 2})\otimes\Gamma(L^{\otimes 2})\\ {\scriptstyle 2_A^*\otimes 2_A^*}\downarrow & & \downarrow{\scriptstyle \xi^*}\\ \Gamma(2_A^*L)\otimes\Gamma(2_A^*L) & \xrightarrow{\simeq} & \Gamma(L^{\otimes 4})\otimes\Gamma(L^{\otimes 4}).\end{array}$$

Let V_α be a vector subspace of $\Gamma(2_A^*L)$ generated by $e^*(s)2_A^*(s')$ where s, $s'\in\Gamma(A, L\otimes P_\alpha)$ and $e^*\colon\Gamma(A, L\otimes P_\alpha)\to k$ is the evaluation map defined by the origine 0 of A for any $\alpha\in\hat{A}_2$. As the canonical map $\Gamma(A, L^{\otimes 2})\otimes\Gamma(A, L^{\otimes 2})\to\Gamma(A, L^{\otimes 4})$ is obtained by

$$\Gamma(L^{\otimes 2})\otimes\Gamma(L^{\otimes 2})\xrightarrow{\xi^*}\Gamma(L^{\otimes 4})\otimes\Gamma(L^{\otimes 4})\xrightarrow{e^*\otimes id}\Gamma(L^{\otimes 4})$$

where $e^*\colon\Gamma(A, L^{\otimes 4})\to k$ is the evaluation map defined by the origine 0 of A, the image of $\Gamma(A, L^{\otimes 2})\otimes\Gamma(A, L^{\otimes 2})\to\Gamma(A, L^{\otimes 4})\simeq\Gamma(A, 2_A^*L)$ is $\sum_{\alpha\in\hat{A}_2}V_\alpha$ by Lemma 2 and the above diagram because e^* satisfies that $e^*(2_A^*s)=e^*(s)$. Hence the canonical map $\Gamma(A, L^{\otimes 2})\otimes\Gamma(A, L^{\otimes 2})\to\Gamma(A, L^{\otimes 4})$ is surjective if and only if $V_\alpha=2_A^*\Gamma(A, L\otimes P_\alpha)$ for any $\alpha\in\hat{A}_2$ because V_α is contained in $2_A^*\Gamma(A, L\otimes P_\alpha)$. If there exists an $s\in\Gamma(A, L\otimes P_\alpha)$ such that $e^*(s)\neq 0$, then it is clear that $V_\alpha=2_A^*\Gamma(A, L\otimes P_\alpha)$. Moreover if any $s\in\Gamma(A, L\otimes P_\alpha)$ satisfies that $e^*(s)=0$, then $V_\alpha=\{0\}\neq 2_A^*\Gamma(A, L\otimes P_\alpha)$. Hence $V_\alpha=2_A^*\Gamma(A, L\otimes P_\alpha)$ if and only if the origine 0 of A is not contained in $\mathrm{Bs}\,|L\otimes P_\alpha|$. Therefore $L^{\otimes 2}$ is normally generated if and only if the origine 0 of A is not contained in $\mathrm{Bs}\,|L\otimes P_\alpha|$ for any $\alpha\in\hat{A}_2$.

Corollary. *If L is an ample line bundle on A and L is base point free, then $L^{\otimes 2}$ is normally generated.*

References

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