30. A Note on the Abstract Cauchy-Kowalewski Theorem

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The purpose of this note is to give a simplified proof and an extention of the nonlinear Cauchy-Kowalewski theorem established by Ovsjannikov [5], Nirenberg [3], Nishida [4] and Kano-Nishida [2] (Appendice). The formulation is generalized, and we need only the contraction mapping principle in the proofs. (See also [1] Appendix C.)

Let $\{X_\rho\,;0\leq\rho\leq\rho_0\}$ be a Banach scale so that $X_\rho\subset X_{\rho'}$ and $|\ \ |_\rho\geq|\ \ |_{\rho'}$ for any $\rho_0\geq\rho\geq\rho'\geq 0$, where $|\ \ |_\rho$ denotes the norm of X_ρ. Consider the equation

(1) $$u(t)=F(t,u(\cdot)),\qquad 0\leq t\leq T.$$

To state the assumptions on F, we introduce some notations. Let $X_{\rho,t}$ be the space of continuous functions $f(s)$ of $s\in[0,t]$ with values in the Banach space X_ρ, which is equipped with the norm

(2) $$|f|_{\rho,t}=\sup_{0\leq s\leq t}|f(s)|_\rho.$$

We also put $X_{\rho,t}(R)=\{f\in X_{\rho,t}\,;|f|_{\rho,t}\leq R\}$.

We state the assumptions on F:

(F.1) There exist constants $R>0$ and $\gamma_0>0$ such that for any $u\in X_{\rho,\tau}(R)$ $F(t,u(\cdot))$ is an $X_{\rho'}$-valued continuous function on $[0,\tau]$ if $0\leq\rho'<\rho\leq\rho_0-\gamma_0\tau$.

(F.2) For $\rho'<\rho(s)\leq\rho\leq\rho_0-\gamma_0\tau$ and $0<\tau\leq T$, F satisfies the following inequality (3) for any $u,v\in X_{\rho,\tau}(R)$:

(3) $$|F(t,u(\cdot))-F(t,v(\cdot))|_{\rho'}\leq\int_0^t C|u(s)-v(s)|_{\rho(s)}/(\rho(s)-\rho')ds,$$

where C is a constant independent of $t,\tau,u,v,\rho,\rho(s)$ or ρ'.

(F.3) For $0<\tau\leq T$ and $\rho\leq\rho_0-\gamma_0\tau$, $F(t,0)$ is continuous in $X_{\rho,\tau}$ and satisfies

(4) $$|F(t,0)|_{\rho_0-\gamma_0 t}\leq R_0<R.$$

For later use we introduce two Banach spaces $Y_{\rho,\gamma}$ and Z_γ of X_ρ-valued continuous functions, by indicating the norms (the range of t being omitted without confusion):

(5) $$\|u\|_{\rho,\gamma}=\sup_{t\geq 0}|u(t)|_{\rho-\gamma t},$$

(6) $$\|u\|_\gamma=\sup_{0\leq\gamma t\leq\rho_0-\rho}|u(t)|_\rho\,\varphi(\gamma t/(\rho_0-\rho)),$$

where $\varphi(t)=(1-t)e^{-t}$. By $Y_{\rho,\gamma}(R)$ we denote the subset $\{f\in Y_{\rho,\gamma}\,;\|f\|_{\rho,\gamma}\leq R\}$. Clearly we have the following:

(7) $\varphi(t)$ is monotone decreasing in $[0,1]$,

(8) $1-\varphi(t)>t\qquad$ for $0<t<1$,

(9) $\qquad \varphi(s)-\varphi(t)\geq e^{-1}(t-s) \qquad$ for $0\leq s\leq t\leq 1$,

(10) $\qquad \|\ \|_\gamma \leq \|\ \|_{\gamma'} \qquad$ for $\gamma\geq\gamma'>0$,

(11) $\qquad \|\ \|_\gamma \leq \|\ \|_{\rho_0,\gamma} \leq (1-\gamma'/\gamma)^{-1}e\|\ \|_{\gamma'} \qquad$ for $\gamma>\gamma'$.

Theorem 1. *Under the assumptions* (F.1), (F.2) *and* (F.3) *there is a constant $\gamma>\gamma_0$ such that there exists a unique solution of* (1) *in* $Y_{\rho_0,\gamma}(R)\cap X_{\rho_0-\gamma\tau,\tau}$ *for* $\tau\in(0,\min\{T,\rho_0/\gamma\}]$.

Consider an equation of extended type:

(12) $$u(t)=F(t,u(\cdot))+\int_0^t E(t,s)G(s,u(\cdot))ds, \qquad 0\leq t\leq T.$$

We assume that F satisfies (F.1)–(F.3) and

(G.1) $G(t,u(\cdot))$ satisfies the same condition as (F.1),

(G.2) for $\rho'<\rho(s)\leq\rho\leq\rho_0-\gamma_0 t$ there holds

(13) $$|G(t,u)-G(t,v)|_{\rho'} \leq B|u(t)-v(t)|_\rho/(\rho-\rho')+B'\int_0^t |u(s)-v(s)|_{\rho(s)}/(\rho(s)-\rho')ds,$$

where B and B' are independent of $t, \tau, u, v, \rho, \rho(s)$ or ρ',

(G.3) for $0<\tau\leq T$ and $\rho\leq\rho_0-\gamma_0\tau$, $G(t,0)$ is continuous in $X_{\rho,\tau}$ and satisfies

(14) $$|G(t,0)|_{\rho_0-\gamma_0 t}\leq R_1.$$

We also assume the linear operator $E(t,s)$ satisfies:

(E) For any $u\in X_\rho$ $E(t,s)u$ is continuous on $\Delta_T=\{(t,s): 0\leq s\leq t\leq T\}$ with values in $X_{\rho'}$ if $\rho'\leq\rho-\beta(t-s)$ with some $\beta\geq 0$ and $\rho<\rho_0-\gamma_0 t$, and there holds

(15) $$|E(t,s)u|_{\rho-\beta(t-s)}\leq A|u|_\rho,$$

where the constant A does not depend on t, s, ρ or ρ'.

Theorem 2. *Under the assumptions* (F.1)–(F.3), (G.1)–(G.3) *and* (E) *there exists a $\gamma>\max(\gamma_0,\beta e)$ such that there is a unique solution $u(t)$ of* (12) *in $Y_{\rho_0,\gamma}(R)\cap X_{\rho_0-\gamma\tau,\tau}(R)$ for any τ, $0<\tau\leq\min(T,\rho_0/\gamma)$.*

Proof of Theorem 1. We define a mapping H from $Z_\gamma\cap X_{\rho,\tau}(R)$ into Z_γ by

(16) $$Hu(t)=F(t,u(\cdot)).$$

Then we have

(17) $$\begin{aligned}|Hu(t)-Hv(t)|_\rho&=|F(t,u(\cdot))-F(t,v(\cdot))|_\rho\\ &\leq\int_0^t C|u(s)-v(s)|_{\rho(s)}(\rho(s)-\rho)^{-1}ds\\ &\leq C\|u-v\|_\gamma\int_0^t \varphi(\gamma s/(\rho_0-\rho(s)))^{-1}(\rho(s)-\rho)^{-1}ds.\end{aligned}$$

We determine $\rho(s)$ by

(18) $$\rho(s)-\rho=(\rho_0-\rho)\varphi(\gamma s/(\rho_0-\rho)).$$

We can take this $\rho(s)$, since (18) implies

(19) $$\begin{aligned}\rho_0-\rho(s)&=(\rho_0-\rho)\{1-\varphi(\gamma s/(\rho_0-\rho))\}\\ &>(\rho_0-\rho)\{\gamma s/(\rho_0-\rho)\}=\gamma s.\end{aligned}$$

Calculating the integrand of (17), we obtain

$$\begin{aligned}&\varphi(\gamma s/(\rho_0-\rho(s)))^{-1}(\rho_0-\rho(s))^{-1}\\ &\qquad\leq 2e(\rho_0-\rho)(\rho_0-\rho-\gamma s)^{-2}e^{\gamma s/(\rho_0-\rho)}.\end{aligned}$$

Here we have used the equality: $1-e^{-x}=xe^{-\theta x}$, $0<\theta<1$, with $x=\gamma s/(\rho_0-\rho)$. Hence we have

$$|Hu(t)-Hv(t)|_\rho\, e^{-\gamma t/(\rho_0-\rho)}\leq C\,\|u-v\|_\gamma \int_0^t 2e(\rho_0-\rho)(\rho_0-\rho-\gamma s)^{-2}ds$$
$$\leq 2Ce\,\|u-v\|_\gamma\,(\rho_0-\rho)\gamma^{-1}(\rho_0-\rho-\gamma t)^{-1}.$$

This implies

$$\|Hu-Hv\|_\gamma\leq(2Ce/\gamma)\,\|u-v\|_\gamma. \tag{20}$$

We choose γ satisfying

$$\gamma\geq\max\,(4\gamma_0/3,\, 8Ce). \tag{21}$$

Then we define an approximating sequence $u_n(t)$ and an associated sequence γ_n by

$$u_0=F(t,0),\qquad u_{n+1}=Hu_n\quad(n\geq 1), \tag{22}$$
$$\gamma_n=\gamma(1-2^{-1-n}),\quad \text{i.e.}\quad \gamma-\gamma_n=\gamma 2^{-n-1}, \tag{23}$$

Clearly it follows (cf. (10) and (11))

$$\|u_{n+1}-u_n\|_{\gamma_n}\leq(2Ce/\gamma_n)\,\|u_n-u_{n-1}\|_{\gamma_n}, \tag{24}$$
$$\|u_{n+1}-u_n\|_{\rho_0,\gamma}\leq(1-\gamma_n/\gamma)^{-1}e\,\|u_{n+1}-u_n\|_{\gamma_n} = 2^{n+1}e\,\|u_{n+1}-u_n\|_{\gamma_n}. \tag{25}$$

These imply

$$\|u_{n+1}-u_n\|_{\rho_0,\gamma}\leq 2^{n+1}e(2Ce)^n\gamma_n^{-n}\,\|u_1-u_0\|_{\gamma_n} \leq 8/3e(4Ce/\gamma)^n\,\|u_1-u_0\|_{\gamma_1}. \tag{26}$$

The assumptions (F.2) and (F.3) imply

$$\|u_1-u_0\|_{\gamma_1}=\|F(\cdot,u_0)-F(\cdot,0)\|_{\gamma_1}\leq(2Ce/\gamma_1)\,\|u_0\|_{\gamma_1}, \tag{27}$$
$$\|u_0\|_{\gamma_1}\leq\|u_0\|_{\rho_0,\gamma_0}\leq R_0. \tag{28}$$

Hence we have

$$\|u_{n+1}-u_n\|_{\rho_0,\gamma}\leq 16/9e(4Ce/\gamma)^{n+1}R_0, \tag{29}$$
$$\|u_{n+1}\|_{\rho_0,\gamma}\{1+4e(4Ce/\gamma)\}R_0. \tag{30}$$

If we choose γ satisfying (21) and

$$\gamma\geq 16Ce^2R_0/(R-R_0), \tag{31}$$

then we obtain

$$\|u_n\|_{\rho_0,\gamma}\leq R,\qquad n\geq 0, \tag{32}$$

which shows $u_n\in Y_{\rho_0,\gamma}(R)$, i.e. $u_n(t)\in X_{\rho_0-\gamma\tau,\tau}(R)$. Thus Hu_n is well-defined. The estimate (29) also implies that $u_n(t)$ converges in $Y_{\rho_0,\gamma}(R)$ and in $X_{\rho_0-\gamma\tau,\tau}(R)$. The limit $u(t)$ is a solution of (1). The uniqueness of the solution in $Y_{\rho,\gamma}$ $(\rho\leq\rho_0)$ is obtained from (20).

We note that the estimate of γ is given by

$$\gamma=\max\,\{4\gamma_0/3,\, 8Ce,\, 16Ce^2R_0/(R-R_0)\}. \tag{Γ.1}$$

Proof of Theorem 2. We define mappings K and L from $Z_\gamma\cap X_{\rho,\tau}(R)$ into Z_γ by

$$Ku(t)=\int_0^t E(t,s)G(s,u(\cdot))ds,\qquad Lu=Hu+Ku. \tag{33}$$

Then from (G.1)–(G.2) and (E), we have

$$|Ku(t)-Kv(t)|_\rho\leq\int_0^t AB\,|u(s)-v(s)|_{\rho(s)}\,\{\rho(s)-(\rho+\beta(t-s))\}^{-1}ds \tag{34}$$
$$+\int_0^t ds\int_0^s AB'\,|u(r)-v(r)|_{\rho(r)}\,\{\rho(r)-(\rho+\beta(t-s))\}^{-1}ds$$

We determine $\rho(s)$ by (18). From (9) we have with $\gamma \geq 2\beta e$

(37) $$\rho(r)-\rho \geq (\rho_0-\rho)\{\varphi(\gamma r/(\rho_0-\rho))-\varphi(\gamma t/(\rho_0-\rho))\} \geq (\rho_0-\rho)e^{-1}\gamma(t-r)/(\rho_0-\rho) \geq 2\beta(t-r),$$

(38) $$\rho(r)-(\rho+\beta(t-r)) \geq (1/2)\{\rho(r)-\rho\}.$$

Thus the same calculations as those from (19) to (20) give

(39) $$\|Ku-Kv\|_r \leq (2Ae/\gamma)\|u-v\|_r(B+B'\tau), \qquad (\gamma\tau \geq \rho_0),$$

(40) $$\|Lu-Lv\|_r \leq \{2(C+AB+AB'\tau_0)e/\gamma\}\|u-v\|_r.$$

We choose τ_0 and γ satisfying

(Γ.2) $$\gamma \geq \max\{4\gamma_0/3, 8De, 2\beta e, \rho_0/\tau_0, 16De^2R_2/(R-R_2)\},$$
$$R_2=R_0+AR_1\tau_0<R, \qquad D=C+AB+AB'\tau_0.$$

Then, we can complete the proof.

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