1. *Lifting of Local Subdifferentiations and Elliptic Boundary Value Problems on Symmetric Domains. I*

By Takashi SUZUKI*) and Ken'ichi NAGASAKI**)

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§ 1. **Introduction.** In [2], C. V. Coffman has shown the existence of non-radial solutions of

(1.1) $$-\Delta u+u=u^{2N+1},\quad u>0\quad (\text{in } \Omega),\quad u=0\quad (\text{on } \partial\Omega)$$

on the annulus domain $\Omega=\{a<|x|<a+c\}\subset \mathbf{R}^2$ for $N=1,2,\cdots$. Immediately is seen that his method applies to

(1.2) $$-\Delta u=u^p,\quad u>0\quad (\text{in } \Omega),\quad u=0\quad (\text{on } \partial\Omega)$$

on the same domain Ω for $1<p<\infty$, and implies the relation

(1.3) $$\sharp\,\{\text{solutions for (1.2)}\}/\mathcal{G}\longrightarrow +\infty$$

as $a\to\infty$, c being fixed, where $\mathcal{G}$ denotes the group of homeomorphisms induced by rotations of independent variables $x\in\Omega$. The proof consists of two parts.

Namely, let $J(v)=\|\nabla v\|_{L^2}/\|v\|_{L^{p+1}}$ and $K_k=\{v\in H_0^1(\Omega)\,|\,T_k v=v,\ v\geqq 0\}\setminus\{0\}$, where T_k denotes the rotation of independent variables $x=\mathrm{r}e^{i\theta}$ for $k=1,2,\cdots$:

$$(T_k v)(\mathrm{r}e^{i\theta})=v(\mathrm{r}e^{i(\theta+(2\pi/k))}).$$

Further, let $K_\infty=\{v\in H_0^1(\Omega)\,|\,v \text{ is radial, } v\geqq 0\}\setminus\{0\}$. Then, first a theorem due to Z. Nahari [8] assures us that each solution $u=u_k$ $(k=1,2,\cdots,\infty)$ of the local variational problem

(1.4) $$\text{To minimize } J(v)\qquad \text{on } v\in K_k$$

satisfies the equation (1.2) by a suitable stretching transformation. Next, the critical values j_k's $(k=1,2,\cdots,\infty)$:

(1.5) $$j_k=\operatorname{Inf}\{J(v)\,|\,v\in K_k\}$$

are separated as $a\to\infty$ after somewhat technical calculations, which guarantees (1.3).

In the manner of Steiner's symmetrization, B. Kawohl has refined the second part of above proof in [6]. That is, it holds that

(1.6) $$m\,|\,n \text{ with } m<n \text{ implies } j_m<j_n \text{ provided that } j_n<j_\infty.$$

Therefore, (1.3) is reduced to showing that for any $m\in N$, $j_m<j_\infty$ follows in the case that a is sufficiently large for each fixed $c>0$.

Still the first part of the above proof depends heavily on the homogenity property of the nonlinear term $f(u)=u^p$ and seems to be impossible to extend into general cases in its original form. However, in the present paper we shall show that generally, solutions of local variational problem

*) Department of Mathematics, Faculty of Science, University of Tokyo.
**) Department of Mathematics, Faculty of Engineering, Chiba Institute of Technology.

such as (1.4) could solve the equation such as (1.2) derived from global variational problem under some property called invariance. Then, we apply the result to show the existence of second radial solutions for elliptic boundary value problems on annulus domains in $\boldsymbol{R}^n$, for rather wide class of nonlinearities.

In forthcoming papers, non-radial solutions for the eigenvalue problem

(1.7) $$-\Delta u=\lambda e^u \quad (\text{in } \Omega), \quad u=0 \quad (\text{on } \partial\Omega),$$

where $\Omega=\{a<|x|<1\}\subset\boldsymbol{R}^2$ and $\lambda>0$, and positive solutions for eigenvalue problems of variational inequalities, will be discussed.

§ 2. Lifting of local sub-differentiations. Let H be a real Hilbert space, and $\varphi, \psi: H\to(-\infty, +\infty]$ be proper convex lower semi-continuous functionals such that $D(\varphi+\psi)=D(\varphi)\cap D(\psi)\neq\phi$, where $D(\varphi)$ denotes the effective domain of $\varphi: D(\varphi)=\{u\in H\,|\,\varphi(u)<+\infty\}$. We recall that for $f\in H'$ and $u\in H$, the relation $f\in\partial\varphi(u)$ indicates that

$$\varphi(\xi)\geqq\varphi(u)+\langle f, \xi-u\rangle$$

for each $\xi\in H$, where $\langle\ ,\ \rangle$ denotes the pairing between H' and H. Therefore, $\partial\varphi(u)\neq\phi$ implies $u\in D(\varphi)$. For the background of the notion $f\in\partial\varphi(u)$, we refer to Brézis [1].

We are interested in the equivalence of two relations

(2.1) $$f\in\partial(\varphi+\psi)(u)$$

and

(2.2) $$f\in\partial\varphi(u).$$

In fact, we have

Proposition 1. *If there exists a $w\in H$ such that*

(2.3) $$f\in\partial\varphi(w) \quad \textit{and} \quad \psi(w)\leqq\psi(u),$$

then (2.1) *implies* (2.2).

Proposition 2. *In the case of $\psi(u)=\mathrm{Inf}\,\{\psi(\xi)\,|\,\xi\in H\}$,* (2.2) *implies* (2.1).

Proof of Proposition 1. Supposing (2.1) and (2.3), we have

(2.1)′ $$\varphi(w)+\psi(w)\geqq\varphi(u)+\psi(u)+\langle f, w-u\rangle$$

and

(2.3)′ $$\varphi(\xi)\geqq\varphi(w)+\langle f, \xi-w\rangle$$

for each $\xi\in H$. Noting that $\psi(w)\leqq\psi(u)<\infty$ and $\varphi(w)<\infty$, we add these two inequalities to obtain

(2.4) $$\varphi(\xi)\geqq\varphi(u)+\langle f, \xi-u\rangle.$$

Proof of Proposition 2. For each $\xi\in H$, we have

$$\varphi(\xi)+\psi(\xi)\geqq\varphi(\xi)+\psi(u)\geqq\varphi(u)+\langle f, \xi-u\rangle+\psi(u).$$

Let $K\subset H$ be a non-void, closed and convex set, and let $\psi=1_K$ be its indicator: $\psi(\xi)=0$ for $\xi\in K$ and $\psi(\xi)=+\infty$ for $\xi\in H\backslash K$. Then, an immediate consequence of these two propositions is

Theorem 1. *For $f\in H'$ and $u\in H$, the following two properties are equivalent:*

(2.5) $$f\in\partial(\varphi+1_K)(u) \quad \textit{and} \quad f\in\partial\varphi(w) \qquad \textit{for some } w\in K$$

(2.6) $$u\in K \quad \textit{and} \quad f\in\partial\varphi(u).$$

Henceforth, the condition $f\in\partial\varphi(w)$ with some $w\in K$ is called the invariance of f with respect to φ in K. Thus, we obtain a solution u of $f \in \partial\varphi(u)$ in K by way of a solution for $f \in \partial(\varphi+1_K)(u)$, when the invariance property holds.

§ 3. Existence of non-minimal solutions. In this section, we consider the eigenvalue problem

(3.1) $$-\Delta u=\lambda f(u) \quad (\text{in } \Omega), \quad u=0 \quad (\text{on } \partial\Omega)$$

on the annulus domain $\Omega=\{x \mid a<|x|<b\}\subset \boldsymbol{R}^n$ for $0<a<b<\infty$. The nonlinear function $f: \boldsymbol{R}\to[0, +\infty)$ is supposed to be C^2 and satisfy

(f1) $$f(0)>0,\ f'(0)\geqq 0 \quad \text{and} \quad f''(t)>0 \qquad (\text{for } t>0)$$

(f2) $$\lim_{t\to+\infty}\frac{f(t)}{t}=+\infty \quad \text{and} \quad \theta f(t)t\geqq F(t) \qquad (\text{for } t \text{ large enough})$$

with $0<\theta<\frac{1}{2}$, where $F(t)=\int_0^t f(s)ds$.

For $\lambda>0$, let $\mathcal{S}_\lambda=\{u \in C^2(\Omega)\cap C^0(\bar{\Omega}) \mid u \text{ solves } (2.1)\}$. Then, it is known that there exists a $\bar{\lambda}>0$ such that $\mathcal{S}_\lambda=\phi$ for $\lambda>\bar{\lambda}$ and $\mathcal{S}_\lambda\neq\phi$ for $0<\lambda<\bar{\lambda}$. Further, there exists a minimal element $\underline{u}_\lambda$, called the minimal solution, in $\mathcal{S}_\lambda$ for each $\lambda \in (0, \bar{\lambda})$. In other words, the relation $v(x)\geqq\underline{u}_\lambda(x)$ $(x \in \Omega)$ holds for any $v \in \mathcal{S}_\lambda$. These results hold for arbitrary domain $\Omega\subset\boldsymbol{R}^n$ with smooth boundary. For the proofs, see Crandall-Rabinowitz [3] and its references. From minimality, $\underline{u}_\lambda$ becomes radial when $\Omega=\{a<|x|<b\}$.

We can show the following

Theorem 2. *For each $\lambda \in (0, \bar{\lambda})$, $\mathcal{S}_\lambda$ contains a radial element other than $\underline{u}_\lambda$.*

In fact, for general domain $\Omega\subset\boldsymbol{R}^n$, Crandall-Rabinowitz [3] has proved the existence of a non-minimal element of $\mathcal{S}_\lambda$ for each $\lambda \in (0, \bar{\lambda})$, under the restriction that the nonlinearity f is of sub-critical, i.e., $f(t)\leqq C(1+t^p)$ $(t>0)$ for $1<p<(n+2)/(n-2)$ in the case $n>2$ and $f(t)\leqq C\exp(1+t^\alpha)$ $(t>0)$ for $\alpha<2$ in the case $n=2$. However, in case of supercritical nonlinearity, the assertion does not hold in general. In fact, we have a counter example that $\Omega=\{|x|<1\}\subset\boldsymbol{R}^n$ and $f(t)=e^t$ for $n\geqq 3$ ([5], cf. [4]). Our point is that the assertion is true even for super-critical nonlinearities in the case of annulus domains.

Proof of Theorem 2 is carried over by tracing that of [3] in the space of radial functions. That is, we take $K=K_\infty$ in the notation of § 1. Then, the mapping $v \in K\mapsto F(v) \in L^p(\Omega)$ $(1<p<\infty)$ is compact and continuous. Thus, the functional

$$J_\lambda(\omega)=\frac{1}{2}\int_\Omega\{|\nabla(\underline{u}_\lambda+\omega)|^2-|\nabla\underline{u}_\lambda|^2\}-\lambda\int_\Omega\{F(\underline{u}_\lambda+\omega)-F(\underline{u}_\lambda)\}$$

is well-defined on K for $\lambda \in (0, \bar{\lambda})$. If J_λ satisfies

(3.2) $$J'_\lambda(\omega)=0$$

for some $\omega \in K\backslash\{0\}$, then $u=\underline{u}_\lambda+\omega \in K\backslash\{\underline{u}_\lambda\}$ satisfies

(3.3) $$\int_\Omega \nabla u\cdot\nabla\chi=\lambda\int_\Omega f(u)\chi \qquad (\lambda \in K),$$

which reads: $\lambda f(u) \in \partial(\varphi+1_K)(u)$ in

$$H=H_0^1(\Omega) \qquad \text{for } \varphi(v)=\frac{1}{2}\int_\Omega |\nabla v|^2 \ (v \in H).$$

Now, it is easy to see the invariance, that is, the existence of $w \in K$ such that $\lambda f(u) \in \partial\varphi(w)$, i.e., $-\Delta w=\lambda f(u)$ (in Ω) and $w=0$ (on $\partial\Omega$), because $u \in K$ is radial and so is $\lambda f(u)$. Therefore, Theorem 1 implies $\lambda f(u) \in \partial\varphi(u)$, and hence Theorem 2 follows.

To show (3.2) for some $w \in K\backslash\{0\}$, we can make use of the mountain pass lemma in K for J_λ. All we have to do is to verify the conditions in [3] in the Hilbert space K. It is not trivial, but rather straight-forward and we omit the proof here.

Remark. We can derive (3.1) directly from (3.3) by writing the latter in polar coordinate and making use of the regularity property for one-dimensional elliptic operators. Such a way of finding radial solutions can be traced back to Kazdan-Warner [7]. Hewever, our way of derivation does not need such regularity, which will get important in the study of other problems such as variational inequalities, for instance.

References

[1] Brézis, H.: Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espace de Hilbert. North-Holland, Amsterdam, London, New York (1973).

[2] Coffman, C. V.: A non-linear boundary value problem with many positive solutions. J. Diff. Eqs., **54**, 429–437 (1984).

[3] Crandall, M. G. and Rabinowitz, P. H.: Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. Arch. Rat. Mech. Anal., **58**, 207–218 (1975).

[4] Gidas, B., Ni, W. N., and Nirenberg, L.: Symmetry and related properties via the maximal principle. Comm. Math. Phys., **68**, 209–243 (1979).

[5] Joseph, J. J. and Lundgren, T. S.: Quasilinear Dirichlet problems driven by positive sources. Arch. Rat. Mech. Anal., **49**, 241–269 (1973).

[6] Kawohl, B.: Rearrangements and Convexity of Level Sets in PPE. Lecture Notes in Math., vol. 1150, Springer, Berlin-Heidelberg-New York-Tokyo, pp. 77–95 (1985).

[7] Kazdan, L. J. and Warner, F. W.: Remarks on some quasilinear elliptic equations. Comm. Pure Appl. Math., **28**, 567–597 (1975).

[8] Nehari, Z.: On a nonlinear differential equation arising in nuclear physics. Proc. Roy. Irish Acad., **62**, 117–135 (1963).