23. On Traces of Hecke Operators on the Spaces of Cusp Forms of Half-integral Weight

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The space of cusp forms of half-integral weight, besides the space of those of integral weight, and Hecke operators on these spaces are well-known objects of arithmetical study of automorphic forms. After S. Niwa [2] had given some explicit relations between traces of these operators, W. Kohnen [1] gave further relation of similar type. In this paper, we shall give more general relations including these results. Details will appear in [3].

Preliminaries.

(a) **General notations.** Let k be a positive integer. If $z \in \boldsymbol{C}$ and $x \in \boldsymbol{C}$, we put $z^x = \exp(x \cdot \log(z))$ with $\log(z) = \log(|z|) + \sqrt{-1}\arg(z)$, $\arg(z)$ being determined by $-\pi < \arg(z) \leq \pi$. For $z \in \boldsymbol{C}$, we put

$$e(z) = \exp(2\pi\sqrt{-1}\, z).$$

Let $\mathfrak{H}$ be the complex upper half plane. For a complex-valued function $f(z)$ on $\mathfrak{H}$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\boldsymbol{R})$, $\gamma = \begin{pmatrix} u & v \\ w & x \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathfrak{H}$, we define functions $J(\alpha, z)$, $j(\gamma, z)$ and $f|[\alpha]_k(z)$ on $\mathfrak{H}$ by: $J(\alpha, z) = cz + d$, $j(\gamma, z) = \left(\frac{-1}{x}\right)^{-1/2} \times \left(\frac{w}{x}\right)(wz + x)^{1/2}$ and $f|[\alpha]_k(z) = (\det \alpha)^{k/2} J(\alpha, z)^{-k} f(\alpha z)$.

(b) **Modular forms of integral weight.** Let N be a positive integer. By $S(2k, N)$, we denote the space of all holomorphic cusp forms of weight $2k$ with the trivial character on the group $\Gamma = \Gamma_0(N)$.

Let $\alpha \in GL_2^+(\boldsymbol{R})$. If Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable, we define a linear operator $[\Gamma\alpha\Gamma]_{2k}$ on $S(2k, N)$ by:

$$f|[\Gamma\alpha\Gamma]_{2k} = (\det \alpha)^{k-1} \textstyle\sum_{\alpha_i} f|[\alpha_i]_{2k},$$

where α_i runs over a system of representatives for $\Gamma\backslash\Gamma\alpha\Gamma$.

For a natural number n with $(n, N) = 1$, we put

$$T_{2k,N}(n) = \textstyle\sum_{ad=n} \left[\Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma\right]_{2k},$$

where the sum is extended over all pairs of integers (a, d) such that $a, d > 0$, $a|d$ and $ad = n$. Moreover, let L_0 be a positive divisor of N such that $(L_0, N/L_0) = 1$ and that $L_0 \neq 1$. Take any element $\gamma(L_0) \in SL_2(\boldsymbol{Z})$ which satisfies the conditions:

$$\gamma(L_0)\equiv\begin{cases}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} & (\bmod L_0);\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & (\bmod N/L_0).\end{cases}$$

Put $W(L_0)=\gamma(L_0)\begin{pmatrix}L_0 & 0\\ 0 & 1\end{pmatrix}$. Then, $W(L_0)$ is a normalizer of Γ and $[W(L_0)]_{2k}$ induces a C-linear automorphism of order 2 on $S(2k, N)$.

(c) **Modular forms of half-integral weight.** Let N be a positive integer divisible by 4 and χ an even character modulo N such that $\chi^2=1$. Put $\mu=\mathrm{ord}_2(N)$, $M=2^{-\mu}N$ and $\Gamma=\Gamma_0(N)$.

Let $G(k+(1/2))$ be the group consisting of pairs (α,φ), where $\alpha=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2^+(\boldsymbol{R})$ and φ is a holomorphic function on $\mathfrak{H}$ satisfying

$$\varphi(z)=t(\det\alpha)^{-k/2-1/4}J(\alpha,z)^{k+1/2}$$

with $t\in C$ and $|t|=1$. The group law is defined by:

$$(\alpha,\varphi(z))\cdot(\beta,\psi(z))=(\alpha\beta,\varphi(\beta z)\psi(z)).$$

For a complex-valued function f on $\mathfrak{H}$ and $(\alpha,\varphi)\in G(k+(1/2))$, we define a function $f|(\alpha,\varphi)$ on $\mathfrak{H}$ by: $f|(\alpha,\varphi)(z)=\varphi(z)^{-1}f(\alpha z)$. By $\Delta=\Delta_0(N,\chi)_{k+1/2}$, we denote a subgroup of $G(k+(1/2))$ consisting of all pairs (γ,φ), where $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\gamma\in\Gamma$ and $\varphi(z)=\chi(d)j(\gamma,z)^{2k+1}$. We denote by $S(k+(1/2),N,\chi)$ the space of all complex-valued holomorphic functions f on $\mathfrak{H}$ which satisfies $f|\xi=f$ for all $\xi\in\Delta$ and which is holomorphic and vanishes at all cusps of Γ. When $\mu=2$, we define the Kohnen subspace $S(k+(1/2),N,\chi)_K$ by:

$$S\left(k+\frac{1}{2},N,\chi\right)_K=\left\{\begin{array}{l}S\left(k+\dfrac{1}{2},N,\chi\right)\ni f(z)=\sum_{n=1}^{\infty}a(n)e(nz);\\ a(n)=0 \text{ for } \varepsilon(-1)^k n\equiv 2,3 \pmod 4\end{array}\right\}.$$

Here, the constant ε is defined as follows: From the assumptions $\mu=2$ and $\chi^2=1$, we can express $\chi=\left(\frac{M_0}{}\right)$. Then, we put $\varepsilon=\left(\frac{-1}{M_0}\right)$.

Let $\xi\in G(k+(1/2))$. If Δ and $\xi^{-1}\Delta\xi$ are commensurable, we define a linear operator $[\Delta\xi\Delta]_{k+1/2}$ on $S(k+(1/2),N,\chi)$ by:

$$f|[\Delta\xi\Delta]_{k+1/2}=\sum_\eta f|\eta,$$

where η runs over a system of representatives for $\Delta\backslash\Delta\xi\Delta$. Then, for a natural number n with $(n,N)=1$, we put

$$\tilde{T}_{k+1/2,N,\chi}(n^2)=n^{k-3/2}\sum_{ad=n}a\left[\Delta\left(\begin{pmatrix}a^2 & 0\\ 0 & d^2\end{pmatrix},(d/a)^{k+1/2}\right)\Delta\right]_{k+1/2},$$

where the sum is extended over all pairs of integers (a,d) such that $a,d>0$, $a|d$ and $ad=n$. Then, the Kohnen subspace $S(k+(1/2),N,\chi)_K$ is invariant under the action of $\tilde{T}_{k+1/2,N,\chi}(n^2)$. Hence, we can consider the trace of $\tilde{T}_{k+1/2,N,\chi}(n^2)$ on $S(k+(1/2),N,\chi)_K$.

Statement of results. Under the above notations, we have the following relations between traces.

Theorem. *Let N be a positive integer such that $2\leq \mathrm{ord}_2(N)=\mu\leq 4$, and put $M=2^{-\mu}N$. Let χ be an even character modulo N such that $\chi^2=1$ and suppose that the conductor of χ is divisible by 8 if $\mu=4$. Then, we have the following relations* (1.1)–(1.2):

(1.1) *Suppose $k\geq 2$, then, for positive integers n with $(n,N)=1$, we have:*

$$\mathrm{trace}\,(\tilde{T}_{k+1/2,N,\chi}(n^2)|S(k+(1/2),N,\chi))$$
$$=\mathrm{trace}\,(T_{2k,N/2}(n)|S(2k,N/2))$$
$$+\textstyle\sum_1 \Lambda(n,L_0)\,\mathrm{trace}\,([W(L_0)]_{2k}T_{2k,2^{\mu-1}L_0L_1}(n)|S(2k,2^{\mu-1}L_0L_1)).$$

(1.2) *Let k and n be the same as in* (1.1) *and suppose $\mu=2$. Then, we have:*

$$\mathrm{trace}\,(\tilde{T}_{k+1/2,N,\chi}(n^2)|S(k+(1/2),N,\chi)_K)$$
$$=\mathrm{trace}\,(T_{2k,M}(n)|S(2k,M))$$
$$+\textstyle\sum_1 \Lambda(n,L_0)\,\mathrm{trace}\,([W(L_0)]_{2k}T_{2k,L_0L_1}(n)|S(2k,L_0L_1)).$$

Here, L_0 in the sum $\sum_1$ runs over all square integers such that $1<L_0|M$. Put $L_1=M\prod_{p|L_0}p^{-\mathrm{ord}_p(M)}$. The constant $\Lambda(n,L_0)$ is defined as follows:

$$\Lambda(n,L_0)=\textstyle\prod_{p|M}\lambda(p,n;(\mathrm{ord}_p(L_0))/2)$$

with

$$\lambda(p,n;a)=\begin{cases}1, & \text{if } a=0;\\ 1+\left(\dfrac{-n}{p}\right), & \text{if } 1\leq a\leq[(\nu-1)/2];\\ \chi_p(-n), & \text{if } \nu \text{ is even and } a=\nu/2;\end{cases}$$

where $\nu=\mathrm{ord}_p(N)$ and χ_p is the p-component of χ.

Remark. For the case $k=1$, we have also some similar relations (cf. [3] § 3 Theorem).

Supplementary remarks. When N and χ are the same as in the Preliminaries (c), the explicit formulas for traces of the operator $\tilde{T}_{k+1/2,N,\chi}(n^2)$ on $S(k+(1/2),N,\chi)$ and $S(k+(1/2),N,\chi)_K$ are given in [3] § 1 and § 2. However, for the other cases than the above Theorem, no relation is found yet. Finally, by using the above Theorem, we can give the decompositions to the eigen subspaces of $S(k+(1/2),N,\chi)$ and $S(k+(1/2),N,\chi)_K$ in several cases.

References

[1] W. Kohnen: Newforms of half-integral weight. J. reine und angew. Math., band 333, 32–72 (1982).

[2] S. Niwa: On Shimura's trace formula. Nagoya Math. J., **66**, 183–202 (1977).

[3] M. Ueda: The decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators (to appear in J. Math. Kyoto Univ.).