18. On the Existence of Periodic Solutions for Periodic Quasilinear Ordinary Differential Systems

By Seiji SAITO and Minoru YAMAMOTO
Department of Applied Physics, Osaka University

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1987)

1. Introduction. In this paper we deal with the problem of the existence of T-periodic solutions for the T-periodic quasilinear ordinary differential system

(1) $$x' = A(t, x)x + F(t, x)$$

where $A(t, x)$ is a real $n \times n$ matrix continuous in (t, x) and T-periodic in t, and $F(t, x)$ is an $\boldsymbol{R}^n$-valued function continuous in (t, x) and T-periodic in t. We consider the associated linear system

(2) $$x' = B(t)x$$

where $B(t)$ is a real $n \times n$ matrix continuous and T-periodic in t.

Hypothesis $\boldsymbol{H}$. There exist no T-periodic solutions of (2) except for the zero solution.

In [1], A. Lasota and Z. Opial discussed the same problem under some hypothesis corresponding to $\boldsymbol{H}$: for each continuous and T-periodic function $y(\cdot)$, $A(\cdot, y(\cdot)) \in M^*$, where M^* is a compact subset of continuous and T-periodic matrices whose systems satisfy Hypothesis $\boldsymbol{H}$. They required that $F(t, x)$ satisfy the following:

$$\liminf_{c \to \infty} \frac{1}{c} \int_0^T \sup_{\|x\| \leqq c} \|F(t, x)\| \, dt = 0.$$

In [2], A. G. Kartsatos considered the existence of T-periodic solutions of (1) under the conditions that $A(t, x)$ is "sufficiently close" to $B(t)$, the system (2) of which satisfies Hypothesis $\boldsymbol{H}$ and that $F(t, x)$ does not depend on x.

In Main Theorem we give an explicit extent that shows how $A(t, x)$ in (1) is close to $B(t)$ in (2) and we obtain certain conditions of $F(t, x)$ which are weaker than those of [1], [2], respectively.

2. Preliminaries. The symbol $\|\cdot\|$ will denote a norm in $\boldsymbol{R}^n$ and the corresponding norm for $n \times n$ matrices. Let C_T be the space of $\boldsymbol{R}^n$-valued functions continuous in $\boldsymbol{R}^1$ and T-periodic with the supremum norm. Let $C[0, T]$ be the space of $\boldsymbol{R}^n$-valued functions continuous on $[0, T]$ with the supremum norm. Let $M[0, T]$ be the space of real $n \times n$ matrices continuous on $[0, T]$ with the supremum norm

$$\|A\|_\infty = \sup\{\|A(t)\|;\ t \in [0, T]\}.$$

We define a bounded linear operator $U: C[0, T] \to \boldsymbol{R}^n$ by $U(x(\cdot)) = x(0) - x(T)$ with the norm

$$\|U\| = \sup\{\|U(x(\cdot))\|;\ \|x\|_\infty = 1\}.$$

We denote $X_B(\cdot)$ by the fundamental matrix of solutions of (2) satisfying $X_B(0)=I$ where I is the identity matrix. Put $U_B=I-X_B(T)$, for $x_0\in \boldsymbol{R}^n$ we have $U(X_B(\cdot)x_0)=U_Bx_0$. We also put $S_r=\{x\in \boldsymbol{R}^n;\ \|x\|\leqq r\}$ and $\boldsymbol{C}_{T,r}=\{y\in C_T;\ \|y\|_\infty\leqq r\}$. Since

$$X_B(t)=I+\int_0^t B(s)X_B(s)ds \quad \text{and} \quad X_B^{-1}(t)=I-\int_0^t X_B^{-1}(s)B(s)ds,$$

applying Gronwall's inequality, we have for any $t\in[0,T]$

$$(3) \qquad \|X_B(t)\|\leqq K, \qquad \|X_B^{-1}(t)\|\leqq K$$

where $K=\exp\left(\int_0^T \|B(s)\|\,ds\right)$.

The following lemmas are well known.

Lemma $\boldsymbol{L_1}$. *Hypothesis* $\boldsymbol{H}$ *is equivalent to* det $U_B\neq 0$. (*See* [3].)

Lemma $\boldsymbol{L_2}$. *If* det $U_B\neq 0$, *then we can choose a positive constant* ρ $(0<\rho<1)$ *satisfying*

$$(4) \qquad \|U_B^{-1}\|\leqq 1/\rho.$$

Suppose that Hypothesis $\boldsymbol{H}$ holds, then there exists ρ in (4) from $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$. Furthermore we assume that positive constants δ, R and functions $A(t,x)$, $F(t,x)$ satisfy the following inequalities:

$$(5) \qquad K^3\delta T\exp(K^2\delta T)\leqq \rho/\{2\|U_B^{-1}\|\}$$

$$(6) \qquad R\leqq \rho(1-\rho)/[KT\exp(\delta T)\{2K^2\exp(2\delta T)+\rho(1-\rho)\}]$$

$$(7) \qquad \|A(t,x)-B(t)\|\leqq\delta \qquad (t\in \boldsymbol{R}^1,\ x\in S_r)$$

$$(8) \qquad \int_0^T \|F(t,x)\|\,dt\leqq rRT \qquad (x\in S_r).$$

3. Main result. We consider the linear nonhomogeneous system of T-periodic differential equations

$$(9) \qquad x'=A(t,y(t))x+F(t,y(t)) \qquad (y\in \boldsymbol{C}_{T,r})$$

together with a boundary condition

$$(10) \qquad U(x)=0 \qquad (x\in C[0,T]).$$

Let $X_y(\cdot)$ be the fundamental matrix of solutions for the homogeneous system corresponding to (9) satisfying $X_y(0)=I$. Put $U_y=I-X_y(T)$, we obtain $U(X_y(\cdot)x_0)=U_yx_0$ for $x_0\in \boldsymbol{R}^n$.

Theorem. *If* (5)–(8) *are satisfied, then for each* y *in* $\boldsymbol{C}_{T,r}$ *there exists the inverse of* U_y *satisfying*

$$(11) \qquad \|U_y^{-1}\|\leqq 1/\{\rho(1-\rho)\}$$

under which Hypothesis $\boldsymbol{H}$ *holds. Moreover the problem* ((9), (10)) *has one and only one solution in* $\boldsymbol{C}_{T,r}$.

Proof of Theorem. By the variation of parameters formula we have

$$X_y(t)=X_B(t)+X_B(t)\int_0^t X_B^{-1}(s)\{A(s,y(s))-B(s)\}X_y(s)\,ds.$$

Then by (3) and (7) we obtain for $t\in[0,T]$

$$\begin{aligned}\|X_y(t)-X_B(t)\| &\leqq \|X_B(t)\|\int_0^t \|X_B^{-1}(s)\|\,\|A(s,y(s))-B(s)\|\,\{\|X_y(s)-X_B(s)\|+\|X_B(s)\|\}ds\\ &\leqq K^2\delta\int_0^t \|X_y(s)-X_B(s)\|\,ds+K^3\delta T.\end{aligned}$$

By (5), applying Gronwall's inequality, we have

(12) $$\|X_y(t)-X_B(t)\| \leqq K^3\delta T \exp(K^2\delta T) \leqq \rho/\{2\|U_B^{-1}\|\}.$$

Then

(13) $$\begin{aligned}\|(U_B-U_y)x_0\| &= \|U(X_B(\cdot)-X_y(\cdot))x_0\| \\ &\leqq 2\|X_B-X_y\|_\infty \|x_0\| \\ &\leqq \rho\|x_0\|/\|U_B^{-1}\|.\end{aligned}$$

By (4) and (13) we obtain for $x_0 \in \boldsymbol{R}^n$

$$\begin{aligned}\rho\|x_0\| &\geqq \|U_B^{-1}\|\|(U_B-U_y)x_0\| \\ &\geqq \|x_0\|-\|U_B^{-1}\|\|U_y x_0\| \\ &\geqq \|x_0\|-\|U_y x_0\|/\rho.\end{aligned}$$

This yields

$$\|U_y x_0\| \geqq \rho(1-\rho)\|x_0\|.$$

Hence U_y has the inverse and (11) holds. Therefore the problem ((9), (10)) has one and only one T-periodic solution x_y :

(14) $$x_y(t) = -U_y^{-1}[U(p_y(\cdot))]+\int_0^t A(s, y(s))x_y(s)ds+\int_0^t F(s, y(s))ds$$

where

$$p_y(t)=X_y(t)\int_0^t X_y^{-1}(s)F(s, y(s))ds.$$

By the same argument used in (3), we obtain $\|X_y(t)\| \leqq K\exp(\delta T)$ and $\|X_y^{-1}(t)\| \leqq K\exp(\delta T)$. This yields

$$\|p_y\|_\infty \leqq rRTK^2\exp(2\delta T).$$

From (14) we obtain for $t \in [0, T]$

$$\|x_y(t)\| \leqq rRT\left\{\frac{2K^2\exp(2\delta T)}{\rho(1-\rho)}+1\right\}+\int_0^t \|A(s, y(s))\|\|x_y(s)\|ds$$

so that, by Gronwall's inequality,

$$\|x_y(t)\| \leqq \frac{rRT\{2K^2\exp(2\delta T)+\rho(1-\rho)\}}{\rho(1-\rho)}\exp\left(\int_0^t \|A(s, y(s))\|ds\right).$$

Thus, by (6), $\|x_y(t)\| \leqq r$. This completes the proof.

Remark. x_y can be expressed by

(15) $$x_y(t) = -X_y(t)U_y^{-1}[U(p_y(\cdot))]+p_y(t).$$

Main Theorem. *If* (5)–(8) *are satisfied, then there exists at least one solution of* (1) *in* $\boldsymbol{C}_{T,r}$, *under which Hypothesis* $\boldsymbol{H}$ *holds.*

Proof of Main Theorem. Define $V: \boldsymbol{C}_{T,r} \rightarrow \boldsymbol{C}_{T,r}$ for $y \in \boldsymbol{C}_{T,r}$ by $(V(y))(t) = x_y(t)$ where x_y is the T-periodic solution of ((9), (10)).

V maps the closed ball $\boldsymbol{C}_{T,r}$ into itself.

Let $y_n \rightarrow y_0$ (n$\rightarrow\infty$) in $\boldsymbol{C}_{T,r}$. In the same way as (12)

$$\|X_{y_n}-X_{y_0}\|_\infty \leqq K^3T\|A(\cdot, y_n(\cdot))-A(\cdot, y_0(\cdot))\|_\infty \exp(K^2T\|A(\cdot, y_n(\cdot))-A(\cdot, y_0(\cdot))\|_\infty)$$

so that

(16) $$X_{y_n} \longrightarrow X_{y_0} \qquad (n\rightarrow\infty)$$

in $M[0, T]$. By the same argument used in (13), we obtain

$$\|(U_{y_n}-U_{y_0})x_0\| \leqq 2\|X_{y_n}-X_{y_0}\|_\infty \|x_0\|.$$

This yields $\|U_{y_n}-U_{y_0}\|\to 0$ $(n\to\infty)$. From the first assertion of Theorem, we have

$$\|U_{y_n}^{-1}-U_{y_0}^{-1}\|\leqq\|U_{y_n}^{-1}\|\|U_{y_0}-U_{y_n}\|\|U_{y_0}^{-1}\|$$
$$\leqq\|U_{y_n}-U_{y_0}\|/\{\rho^2(1-\rho)^2\}.$$

This yields $\|U_{y_n}^{-1}-U_{y_0}^{-1}\|\to 0$ $(n\to\infty)$. From the variation of parameters formula we have

$$X_{y_n}^{-1}(t)-X_{y_0}^{-1}(t)$$
$$=\left\{\int_0^t X_{y_n}^{-1}(s)\{A(s,y_0(s))-A(s,y_n(s))\}X_{y_0}(s)ds\right\}X_{y_0}^{-1}(t).$$

By the same argument used in (16), we obtain $X_{y_n}^{-1}\to X_{y_0}^{-1}$ $(n\to\infty)$ in $M[0,T]$. This implies $p_{y_n}\to p_{y_0}$ $(n\to\infty)$ in $C[0,T]$. Thus, by (15), $V(y_n)\to V(y_0)$ $(n\to\infty)$ in $\boldsymbol{C}_{T,r}$.

It is clear that $V(\boldsymbol{C}_{T,r})$ is uniformly bounded. From (14) it follows that for $y\in\boldsymbol{C}_{T,r}$

$$\|V(y)(t_1)-V(y)(t_2)\|$$
$$\leqq\left|\int_{t_1}^{t_2}\|A(s,y(s))\|rds\right|+\left|\int_{t_1}^{t_2}\|F(s,y(s))\|ds\right|$$
$$\leqq\{(\delta+\|B\|_\infty)r+N\}|t_1-t_2| \qquad (t_1,t_2\in[0,T])$$

where $N=\max\{\|F(t,x)\|;\ t\in[0,T],\ x\in S_r\}$. Consequently, $V(\boldsymbol{C}_{T,r})$ is equicontinuous. By Ascoli-Arzerà theorem $V(\boldsymbol{C}_{T,r})$ is a relatively compact set in $\boldsymbol{C}_{T,r}$.

According to Schauder's fixed point theorem, V has at least one fixed point in $\boldsymbol{C}_{T,r}$. Therefore (1) has at least one solution in $\boldsymbol{C}_{T,r}$, and this completes the proof.

References

[1] A. Lasota and Z. Opial: Sur les solutions périodiques des équations différentielles ordinaires. Ann. Pol. Math., **16**, 69–94 (1964).

[2] A. G. Kartsatos: Advanced Differential Equations. Mariner Publishing Co. Inc., Tampa, Florida (1980).

[3] E. A. Coddington and N. Levinson: Theory of Ordinary Differential Equations. McGraw-Hill, New York (1955).