17. The Vanishing Viscosity Method and a Two-phase Stefan Problem with Nonlinear Flux Condition of Signorini Type

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1. Introduction. This paper is concerned with a two-phase Stefan problem with nonlinear flux condition of the so-called Signorini type. Let Ω be a bounded domain in R^N ($N\geqq 2$) whose boundary consists of two smooth disjoint surfaces Γ_0, Γ_1, and let T be a fixed positive number, $Q=(0,T)\times\Omega$, $\Sigma_0=(0,T)\times\Gamma_0$, and $\Sigma_1=(0,T)\times\Gamma_1$. The problem, denoted by (P), is to find a function $u=u(t,x)$ on Q satisfying

$$u_t-\Delta\beta(u)=0 \qquad \text{in } Q,$$
$$u(0,\cdot)=u_0 \qquad \text{in } \Omega,$$
$$\beta(u)=g_0 \qquad \text{on } \Sigma_0,$$
$$-\frac{\partial\beta(u)}{\partial n}\in\gamma(\beta(u)-g_1) \qquad \text{on } \Sigma_1.$$

Here $\beta: R\to R$ is a given function which vanishes on $[0,1]$, is non-decreasing on R and bi-Lipschitz continuous both on $(-\infty,0]$ and $[1,+\infty)$; γ is a multivalued function from R into R given by $\gamma(r)=0$ for $r>0$, $\gamma(0)=(-\infty,0]$ and $\gamma(r)=\emptyset$ for $r<0$; u_0 is a given initial datum; g_0 and g_1 are given functions on Σ_0 and Σ_1, respectively; $(\partial/\partial n)$ denotes the outward normal derivative. For the data we postulate that

(A1) g_i $(i=0,1)$ is the trace of a function, denoted by g_i again, on Q such that $g_i\in W^{1,2}(0,T;H^1(\Omega))\cap L^\infty(0,T;H^2(\Omega))$, $m_0\leqq g_0\leqq m_0'$, $m_1\geqq g_1\geqq m_1'$ a.e. on Q, where $m_0\leqq m_0'<0$, $m_1\geqq m_1'>0$ are constants.

(A2) (i) $u_0\in L^\infty(\Omega)$, meas. $\{x\in\Omega;0\leqq u_0(x)\leqq 1\}=0$, $v_0=\beta(u_0)\in H^1(\Omega)$; (ii) $v_0=g_0(0,\cdot)$ a.e. on Γ_0, $v_0\geqq g_1(0,\cdot)$ a.e. on Γ_1; (iii) there are constants $\delta>0$, $k_0<0$, $k_1>0$ such that $v_0\leqq k_0$ a.e. on $\Omega_{0,\delta}$ and $v_0\geqq k_1$ a.e. on $\Omega_{1,\delta}$, where

$$\Omega_{i,\delta}=\{x\in\Omega;\ \text{dist.}(x,\Gamma_i)<\delta\}, \qquad i=0,1.$$

In particular, when g_0 and g_1 are independent of time t, problem (P) was treated by Magenes-Verdi-Visintin [6] in the framework of nonlinear contraction semigroups in $L^1(\Omega)$ (cf. Bénilan [1], Crandall [3]), and the solution is unique in the sense of Crandall-Liggett [4]. Also, in case the flux condition is of the form $-(\partial/\partial n)\beta(u)=\gamma(t,x,\beta(u))$, with smooth function $\gamma(t,x,r)$ on $\Sigma_1\times R$, the problem was uniquely solved in variational sense by Niezgodka-Pawlow [7], Visintin [9] and Niezgodka-Pawlow-Visintin [8].

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However, when the boundary flux is governed by a general time-dependent maximal monotone graph $\gamma(t, x, \cdot)$, we have not noticed any results, in particular on the uniqueness of solution. The purpose of the present note is to construct a solution of (P) by the vanishing viscosity method and to show the uniqueness of the solution constructed in such a way.

We use the following notations : $H=L^2(\Omega)$, $X=H^1(\Omega)$, $X_0=\{z\in X\,;\, z=0$ a.e. on $\Gamma_0\}$, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X_0' and X_0, dS denotes the usual surface element on Γ_0, Γ_1, and

$$(u, v)=\int_\Omega uvdx, \qquad a(u, v)=\int_\Omega \nabla u\cdot\nabla vdx.$$

2. Main results. We give a notion of solution to (P) in the variational sense.

Definition 1. A function $u:[0, T]\to H$ is called a weak solution of (P), if it satisfies the following (V1)–(V4) :

(V1) $u\in W^{1,2}(0, T\,;\, X_0')\cap L^\infty(Q)$, $\beta(u)\in W^{1,2}(0, T\,;\, H)\cap L^2(0, T\,;\, X)$;

(V2) $u(0)=u_0$ (in the space H);

(V3) $\beta(u)=g_0$ a.e. on Σ_0;

(V4) there is $f\in L^2(\Sigma_1)$ such that $f\in\gamma(\beta(u)-g_1)$ a.e. on Σ_1, and $\langle u'(t), \zeta\rangle+a(\beta(u(t)), \zeta)+\int_{\Gamma_1} f(t, \cdot)\zeta dS=0$ for any $\zeta\in X_0$ and a.e. $t\in[0, T]$.

It should be remarked that if $\beta(u(t))\in H^2(\Omega_{1,\delta})$ for some $\delta>0$, then $f(t, \cdot)=-(\partial/\partial n)\beta(u(t, \cdot))$ on Γ_1 in (V4).

Now, consider approximations β^ν of β and γ_ε of γ, defined by

$$\beta^\nu(r)=\beta(r)+\nu r, \quad \nu\in(0,1], \quad \gamma_\varepsilon(r)=-(-r)^+/\varepsilon, \quad \varepsilon\in(0,1].$$

Then we denote by $(P)_\varepsilon$ the problem (P) with γ replaced by γ_ε, and by $(P)^\nu$ the problem (P) with β and u_0 replaced by β^ν and $u_0^\nu=(\beta^\nu)^{-1}(v_0)$. The problems $(P)_\varepsilon$, $(P)^\nu$ represent standard approximations to (P) and their weak solutions are defined correspondingly.

By virtue of the results in [7] we know that (i) for each $\varepsilon\in(0,1]$, $(P)_\varepsilon$ has one and only one weak solution, denoted by u_ε; (ii) if $0<\varepsilon<\bar\varepsilon\leqq 1$, then $u_{\bar\varepsilon}\leqq u_\varepsilon$ a.e. on Q. Also, by the results in [5], for each $\nu\in(0,1]$, $(P)^\nu$ has one and only one weak solution in $W^{1,2}(0, T\,;\, H)\cap L^\infty(0, T\,;\, X)$, which is denoted by u^ν.

Definition 2. A function $u:[0, T]\to H$ is called a solution of (P) in the vanishing viscosity sense (in short, a V-solution of (P)), if it is a weak solution of (P) and if there is a sequence $u^{\nu_n}\in W^{1,2}(0, T\,;\, H)\cap L^\infty(0, T\,;\, X)$ of weak solutions of $(P)^{\nu_n}$ such that $u^{\nu_n}\to u$ in the weak* topology of $L^\infty(Q)$ as $n\to+\infty$.

Our main results are stated as follows.

Theorem. *Suppose* (A1) *and* (A2) *hold. Then we have the following statements* :

(a) (P) *has at least one V-solution*;

(b) *any V-solution of* (P) *has the property that* $u\in W^{1,2}(0, T\,;\, L^2(\Omega'))$, $\beta(u)\in L^2(0, T\,;\, H^2(\Omega'))$, *where* $\Omega'=\Omega_{0,\delta}\cup\Omega_{1,\delta}$ *for some* $\delta>0$;

(c) *any V-solution of* (P) *coincides with the limit* u^* *of the weak solutions* u_ε *of* $(\mathrm{P})_\varepsilon$ *as* $\varepsilon \downarrow 0$.

From this theorem it immediately follows that (P) has one and only one V-solution, and the weak solutions u^ν of $(\mathrm{P})^\nu$ converge to the V-solution u of (P) as $\nu \downarrow 0$ in such a way that $u^\nu \to u$ weakly* in $L^\infty(Q)$, $\beta^\nu(u^\nu) \to \beta(u)$ strongly in $L^2(Q)$ and weakly in $L^2(0, T; X)$, $(\partial/\partial n)\beta^\nu(u^\nu) \to (\partial/\partial n)\beta(u)$ weakly in $L^2(\Sigma_1)$ and $\beta^\nu(u^\nu)_t \to \beta(u)_t$ weakly in $L^2(Q)$.

3. Sketch of the proof. In order to obtain some bounds for V-solutions of (P) we consider the approximate problem $(\mathrm{P})^\nu_\varepsilon$ which is the problem (P) with β, γ, u_0 replaced by β^ν, γ_ε, u^ν_0. We denote by u^ν_ε the weak solution of $(\mathrm{P})^\nu_\varepsilon$ for each $\nu \in (0, 1]$ and $\varepsilon \in (0, 1]$. We have the following estimates independent of ν and ε.

(1) $|u^\nu_\varepsilon|_{L^\infty(Q)} \leqq M$, where M is any constant satisfying $|u_0|_{L^\infty(\Omega)} \leqq M$, $\beta(-M) \leqq m_0$ and $\beta(M) \geqq m_1$.

(2) $\beta^\nu(u^\nu_\varepsilon) \leqq -c$ a.e. on $Q_{0,\delta} = (0, T) \times \Omega_{0,\delta}$ and $\beta^\nu(u^\nu_\varepsilon) \geqq c$ a.e. on $Q_{1,\delta} = (0, T) \times \Omega_{1,\delta}$ for some constants $c > 0$ and $\delta > 0$.

(3) $\{\beta^\nu(u^\nu_\varepsilon); 0 < \nu \leqq 1, 0 < \varepsilon \leqq 1\}$ is bounded in $W^{1,2}(0, T; H) \cap L^\infty(0, T; X)$ and in $L^2(0, T; H^2(\Omega'))$, with $\Omega' = \Omega_{0,\delta} \cup \Omega_{1,\delta}$ for some $\delta > 0$, and hence $\{u^\nu_\varepsilon; 0 < \nu \leqq 1, 0 < \varepsilon \leqq 1\}$ is bounded in $W^{1,2}(0, T; L^2(\Omega'))$.

In fact, estimates (1) and (2) are obtained from assumptions (A1), (A2) and the usual comparison results, and (3) is shown by making use of regularity results in Brézis [2; Chapter 1]. Next, by the monotonicity of solutions u^ν_ε with respect to ε we have:

(4) For each $\nu \in (0, 1]$, $u^\nu_\varepsilon \uparrow u^\nu$ strongly in $L^2(Q)$ and weakly in $W^{1,2}(0, T; H)$ as $\varepsilon \downarrow 0$, and $\{u^\nu; 0 < \nu \leqq 1\}$ has the same bounds as (1)–(3).

Besides, by the uniqueness of solution to $(\mathrm{P})_\varepsilon$ and estimates (1)–(3) we see:

(5) For each $\varepsilon \in (0, 1]$, $u^\nu_\varepsilon \to u_\varepsilon$ weakly* in $L^\infty(Q)$, $\beta^\nu(u^\nu_\varepsilon) \to \beta(u_\varepsilon)$ strongly in $L^2(Q)$ and weakly in $L^2(0, T; X)$, $(\partial/\partial n)\beta^\nu(u^\nu_\varepsilon) \to (\partial/\partial n)\beta(u_\varepsilon)$ strongly in $L^2(\Sigma_1)$, and $\beta^\nu(u^\nu_\varepsilon)_t \to \beta(u_\varepsilon)_t$ weakly in $L^2(Q)$ as $\nu \downarrow 0$, and moreover $\{u_\varepsilon; 0 < \varepsilon \leqq 1\}$ has the same bounds as (1)–(3).

Using the facts (1)–(5), we can prove the theorem as follows. Let u^* be the limit of u_ε as $\varepsilon \downarrow 0$. Note that there exists a sequence $\{\nu_n\}$ with $\nu_n \downarrow 0$ (as $n \to \infty$) such that $u^{\nu_n} \to u$ weakly* in $L^\infty(Q)$, $\beta^{\nu_n}(u^{\nu_n}) \to \beta(u)$ strongly in $L^2(Q)$ and weakly in $L^2(0, T; X)$, $(\partial/\partial n)\beta^{\nu_n}(u^{\nu_n}) \to (\partial/\partial n)\beta(u)$ weakly in $L^2(\Sigma_1)$, and $\beta^{\nu_n}(u^{\nu_n})_t \to \beta(u)_t$ weakly in $L^2(Q)$ for some function $u \in L^\infty(Q)$. Then both u^* and u are weak solutions of (P), and by definition u is a V-solution of (P). Moreover, $u^* \leqq u$ a.e. on Q, since $u^{\nu_n}_\varepsilon \leqq u^{\nu_n}$ a.e. on Q. Besides, $\beta(u^*)$, $\beta(u) \in L^2(0, T; H^2(\Omega'))$. Hence by monotonicity arguments $(\partial/\partial n)\beta(u) \leqq (\partial/\partial n)\beta(u^*)$ a.e. on Σ_1, and for the solution ζ of $-\Delta\zeta = 0$ in Ω with $\zeta = 0$ on Γ_0 and $\zeta = 1$ on Γ_1, we observe from (V4) that

$$\langle u'(t) - u^{*\prime}(t), \zeta \rangle - (\beta(u(t)) - \beta(u^*(t)), \Delta\zeta) + \int_{\Gamma_1} (\beta(u(t, \cdot)) - \beta(u^*(t, \cdot))) \frac{\partial \zeta}{\partial n}\, dS - \int_{\Gamma_1} \left(\frac{\partial \beta(u(t, \cdot))}{\partial n} - \frac{\partial \beta(u^*(t, \cdot))}{\partial n} \right) dS = 0$$

for a.e. $t \in [0, T]$. Noting $(\partial/\partial n)\zeta \geqq 0$ on Γ_1, we have

$$\frac{d}{dt}(u(t)-u^*(t), \zeta) = \langle u'(t)-u^{*\prime}(t), \zeta \rangle \leqq 0 \qquad \text{for a.e. } t \in [0, T].$$

Since $\zeta > 0$ and $u(t, \cdot) \geqq u^*(t, \cdot)$ in Ω, this implies $u = u^*$ on Q.

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