16. Applications of Spreading Models to an Equivalence of Summabilities and Growth Rate of Cesàro Means

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0. Introduction. In this paper, we present two applications of Brunel-Sucheston spreading models. One application is to estimate, from above, the growth rate of Cesàro means and the other one is to discuss an equivalence between regular methods of summability. The complete proofs of our results and related ones will appear elsewhere.

Throughout this paper, X denotes a Banach space, N denotes the set of all positive integers, and S_0 denotes the vector space of finite scalar sequences with the canonical unit vector basis $\{e_n\}_n$.

1. Brunel-Sucheston spreading model. We start by explaining the concept of the Brunel-Sucheston spreading model. Let $\{x_n\}_n$ be a bounded sequence with no norm Cauchy subsequence in a Banach space X. Suppose that the limit

$$\lim_{\substack{m\to\infty \\ m\leqq n_1<\cdots<n_k}} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\|$$

exists for all $(a_i)_{i=1}^k$ in S_0. We shall call such a sequence $\{x_n\}_n$ a *BS-sequence* (named after Brunel and Sucheston). Then we can define the nonnegative function Ψ on S_0 by

$$\Psi((a_i)_{i=1}^k) := \lim_{\substack{m\to\infty \\ m\leqq n_1<\cdots<n_k}} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\|.$$

It is known that Ψ defines a norm on S_0 (see [3, p. 296]), hence we shall write $\|\sum_{i=1}^k a_i e_i\|$ in place of $\Psi((a_i)_{i=1}^k)$ for each $(a_i)_{i=1}^k$ in S_0. Let E be the completion of $[S_0, \|\cdot\|]$. We say that $[E, \{e_n\}_n]$ is the *spreading model* of $\{x_n\}_n$. In [3], Brunel and Sucheston proved that every bounded sequence in any Banach space with no norm Cauchy subsequence has a subsequence which is a *BS*-sequence. Then $\{x_n\}_n$ and its spreading model $[E, \{e_n\}_n]$ have the following properties (Spreading Model):

(1) $$\|\sum_{i\in A_1} a_i(e_{2i-1}-e_{2i})\| \leqq \|\sum_{i\in A_2} a_i(e_{2i-1}-e_{2i})\|$$

for each finite subsets A_1, A_2 of N with $A_1\subset A_2$ and $(a_i)_i$ in S_0.

(2) $$\lim_{\substack{m\to\infty \\ m\leqq n_1<\cdots<n_k}} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\| = \left\|\sum_{i=1}^{k} a_i e_i\right\|$$

for every vector $(a_i)_{i=1}^k$ in S_0.

(3) For any $\varepsilon>0$ and k in N there exists an $L(\varepsilon, k)$ in N so that for every $(a_i)_{i=1}^k$ in S_0 and n_i in N $(i=1, 2, \cdots, k)$ with $L(\varepsilon, k)\leqq n_1<n_2<\cdots<n_k$,

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$$(1-\varepsilon)\left\|\sum_{i=1}^{k} a_i e_i\right\| \leqq \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\| \leqq (1+\varepsilon)\left\|\sum_{i=1}^{k} a_i e_i\right\|.$$

Moreover, if in addition, $\{x_n\}_n$ is a weakly null sequence, i.e., weak-$\lim_{n\to\infty} x_n = 0$, then we have

$$(1') \qquad \left\|\sum_{i\in A_1} a_i e_i\right\| \leqq \left\|\sum_{i\in A_2} a_i e_i\right\|$$

for each finite subsets A_1, A_2 of N with $A_1 \subset A_2$ and $(a_i)_i$ in S_0.

For proofs of (1) and (1′), see [2, p. 360], (2) is obvious and (3) is easily checked by an ε-net argument.

By applying the Brunel-Sucheston spreading model, we get the following fundamental result, which is an infinite dimensional version for the property (3) in (Spreading Model).

Theorem 1. *Let $\{x_n\}_n$ be a BS-sequence in X and $[E, \{e_n\}_n]$ be its spreading model. Assume that $\{x_n\}_n$ is a weakly null sequence. Then for any $\varepsilon>0$ and integer $t \geqq 2$ one can select a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ with the following property:*

$$\frac{1}{4}(1-\varepsilon)\left\|\sum_{i=1}^{k} e_i\right\| - 2(\log_t k)\sup_n \|x'_n\| \leqq \left\|\sum_{i=1}^{k} \theta_i x'_{n_i}\right\|$$

and

$$\left\|\sum_{i=1}^{k} a_i x'_{n_i}\right\| \leqq 4(1+\varepsilon)\left\|\sum_{i=1}^{k} e_i\right\| + 3(\log_t k)\sup_n \|x'_n\|$$

for all k, n_i in N ($i=1,2,\cdots,k$) with $n_1<n_2<\cdots<n_k$, $(a_i)_{i=1}^k$, $(\theta_i)_{i=1}^k$ with $|a_i| \leqq 1$, $|\theta_i|=1$ ($i=1,2,\cdots,k$).

By using Theorem 1, we can get an "alternative" theorem concerning weakly null sequences.

Theorem 2. *For every weakly null sequence $\{x_n\}_n$ in X one can extract a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that either*

$$(1) \qquad \lim_{k\to\infty} \sup_{\substack{n_1<\cdots<n_k \\ |a_i|\leqq 1}} \left\|\frac{1}{k}\sum_{i=1}^{k} a_i x'_{n_i}\right\| = 0$$

or

$$(2) \qquad \inf_k \inf_{\substack{n_1<\cdots<n_k \\ |\theta_i|=1}} \left\|\frac{1}{k}\sum_{i=1}^{k} \theta_i x'_{n_i}\right\| > 0.$$

2. Growth rate of Cesàro means. In [1], Banach and Saks proved that $L_p[0,1]$ ($1<p<\infty$) has the so-called *Banach-Saks property* by actually showing the following:

Each weakly null sequence $\{x_n\}_n$ in $L_p[0,1]$ has a subsequence $\{x'_n\}_n$ which satisfies

$$\left\|\sum_{i=1}^{k} x'_i\right\|_p = \begin{cases} O(k^{1/p}) & \text{if } 1<p\leqq 2 \\ O(k^{1/2}) & \text{if } 2\leqq p<\infty. \end{cases}$$

Recall that a Banach space X is of *type p* with $1<p\leqq 2$, if there is a constant $M \geqq 1$ such that for every finite set of vectors $\{x_i\}_{i=1}^k$ in X we have

$$\operatorname*{Average}_{\theta_i=\pm 1}\left\|\sum_{i=1}^{k} \theta_i x_i\right\| \leqq M\left(\sum_{i=1}^{k} \|x_i\|^p\right)^{1/p}.$$

It is known that $L_p[0,1]$ is of type $\min(2,p)$ (see [5, p. 73]).

We can show the following theorem which is a natural generalization of the result of Banach and Saks. Our method of the proof is completely different from that of Banach and Saks.

Theorem 3. *Let X be a Banach space of type p with $1<p\leqq 2$ and M be a type p constant of X. Then for each weakly null sequence $\{x_n\}_n$ in X one can extract a subsequence $\{x'_n\}_n$ so that*

$$\sup_{|a_i|\leqq 1}\left\|\sum_{i=1}^{k} a_i x'_{n_i}\right\|\leqq 78M\sup_n\|x'_n\|\,k^{1/p}$$

for every k, n_i in N $(i=1,2,\cdots,k)$ with $n_1<n_2<\cdots<n_k$.

Sketch of Proof. Let $\{x_n\}_n$ be a weakly null sequence in X. We may assume that $\{x_n\}_n$ has no norm convergent subsequence. By Theorem 1, $\{x_n\}_n$ has a subsequence $\{x'_n\}_n$ which is a BS-sequence with its spreading model $[E,\{e_n\}_n]$ and satisfies

$$\frac{1}{5}\left\|\sum_{i=1}^{k} e_i\right\|-2(\log_3 k)\sup_n\|x'_n\|\leqq\inf_{|\theta_i|=1}\left\|\sum_{i=1}^{k}\theta_i x'_{n_i}\right\|$$

and

$$\sup_{|a_i|\leqq 1}\left\|\sum_{i=1}^{k} a_i x'_{n_i}\right\|\leqq 5\left\|\sum_{i=1}^{k} e_i\right\|+3(\log_3 k)\sup_n\|x'_n\|$$

for each k, n_i in N $(i=1,2,\cdots,k)$ with $n_1<n_2<\cdots<n_k$.

By using the first inequality, we have

$$\begin{aligned}\frac{1}{5}\left\|\sum_{i=1}^{k} e_i\right\|-2(\log_3 k)\sup_n\|x'_n\| &\leqq \underset{\theta_i=\pm 1}{\mathrm{Average}}\left\|\sum_{i=1}^{k}\theta_i x'_i\right\|\\ &\leqq M\Big(\sum_{i=1}^{k}\|x'_i\|^p\Big)^{1/p}\\ &\leqq M\sup_n\|x'_n\|\,k^{1/p},\end{aligned}$$

hence by the second inequality above, we obtain

$$\begin{aligned}\sup_{|a_i|\leqq 1}\left\|\sum_{i=1}^{k} a_i x'_{n_i}\right\| &\leqq\{5(5M\,k^{1/p}+10\,k^{1/p})+3\,k^{1/p}\}\sup_n\|x'_n\|\\ &\leqq 78M\sup_n\|x'_n\|\,k^{1/p}\end{aligned}$$

for all k, n_i in N $(i=1,2,\cdots,k)$ with $n_1<n_2<\cdots<n_k$.

3. An equivalence of regular methods of summability. An infinite matrix $(a_{n,m})$ is called a *regular method of summability* (see [4, p. 96]), if the following conditions hold:

(1) $$\sup_n\sum_{m=1}^{\infty}|a_{n,m}|<\infty,$$

(2) $$\lim_{n\to\infty}\sum_{m=1}^{\infty}a_{n,m}=1,$$

(3) $$\lim_{n\to\infty}a_{n,m}=0\qquad(m\geqq 1).$$

An interesting method of summability is that of Cesàro's:

$$C:=(c_{n,m})\text{ with }c_{n,m}:=1/n\ (1\leqq m\leqq n)\text{ and }c_{n,m}:=0\ (1\leqq n<m).$$

On the other hand, the most trivial one is the identity summability:

$$I:=(\delta_{n,m})\text{ with }\delta_{n,m}:=1\ (n=m)\text{ and }\delta_{n,m}:=0\ (n\neq m).$$

For a regular method of summability $A=(a_{n,m})$, a bounded sequence $\{x_n\}_n$ in X is called *A-summable* to an element x_0 in X if

$$\lim_{n\to\infty}\left\|\sum_{m=1}^{\infty} a_{n,m}x_m - x_0\right\| = 0.$$

Now we introduce a stronger notion of summability as follows. A bounded sequence $\{x_n\}_n$ in X is said to be *completely A-summable* to x_0 if each subsequence of $\{x_n\}_n$ is A-summable to x_0. In terms of this complete summability, we define an equivalence relation among regular methods of summability as follows. For regular methods of summability A and B, A is said to be *stronger* than B if the following condition is satisfied:

If a bounded sequence $\{x_n\}_n$ in an arbitrary Banach space is completely A-summable to x_0, then there is a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that $\{x'_n\}_n$ is completely B-summable to x_0.

We say that A is *equivalent* to B if A is stronger than B and B is stronger than A.

With respect to this equivalence, we have the following:

Theorem 4. *Every regular method of summability $A=(a_{n,m})$ is equivalent to either Cesàro summability or the identity summability according as*

$$\lim_{n\to\infty}(\sup_m |a_{n,m}|)=0 \quad or \quad \limsup_{n\to\infty}(\sup_m |a_{n,m}|)>0.$$

This result shows that Cesàro summability is, in a sense, the most fundamental summability. A part of our proof of the above theorem depends upon Theorem 2.

References

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