2. $A \geqq B \geqq 0$ iff $(B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$ for $r \geqq 0, p \geqq 0, q \geqq 1$ with $(1+2r)q \geqq p+2r$

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A capital letter means a bounded linear opeartor on a Hilbert space. An operator T is said to be positive in case $(Tx, x) \geqq 0$ for every x in a Hilbert space.

What functions preserve the ordering of positive operators? In other words, what must f satisfy so that

$$A \geqq B \geqq 0 \quad \text{implies} \quad f(A) \geqq f(B)?$$

A function f is said to be operator monotone if f satisfies the property stated above. This problem was first studied by K. Löwner, who had given a complete description of operator monotone functions. Also he had shown the following result in [7].

Theorem A. *If A and B are bounded positive operators on a Hilbert space such that $A \geqq B \geqq 0$, then $A^\alpha \geqq B^\alpha$ for each α in the interval* $[0, 1]$.

This theorem had been also shown by E. Heinz [4] and also T. Kato [5] had given a shorter proof. Recently two simple proofs have been shown by Au-Yeung [1] and Man Kam Kwong [6]. An elegant and simple proof based on C^*-algebra theory of Theorem A has been shown in [8].

Nevertheless it is well known that $A \geqq B \geqq 0$ does not always assure $A^2 \geqq B^2$ in general. We know almost no knowledge except both commutative case and operator monotone function case.

The purpose of this paper is to announce early "order preserving inequalities" on A and B in case $A \geqq B \geqq 0$, that is, we have found two order preserving functions $f(X)$ and $g(Y)$ under suitable and agreeable additional conditions. We explain these functions in Remark 1 and also these conditions in Remark 3.

Theorem 1. *If $A \geqq B \geqq 0$, then for each $r \geqq 0$*

(i) $$(B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$$

and

(ii) $$A^{(p+2r)/q} \geqq (A^r B^p A^r)^{1/q}$$

hold for each p and q such that $p \geqq 0$, $q \geqq 1$ and $(1+2r)q \geqq p+2r$.

Corollary 1. *If $A \geqq B \geqq 0$, then for each $r \geqq 0$*

(i) $$(B^r A^p B^r)^{(1+2r)/(p+2r)} \geqq B^{1+2r}$$

and

(ii) $$A^{1+2r} \geqq (A^r B^p A^r)^{(1+2r)/(p+2r)}$$

hold for each $p \geqq 1$.

Corollary 2. *If $A \geqq B \geqq 0$, then for each $r \geqq 0$*

(i) $$(B^rA^pB^r)^{1/p}\geqq B^{(p+2r)/p}$$
and
(ii) $$A^{(p+2r)/p}\geqq(A^rB^pA^r)^{1/p}$$
hold for each $p\geqq 1$.

Remark 1. We consider two operator functions f and g depending on A and B such that

$$f(X)=(B^rXB^r)^{1/q} \quad\text{and}\quad g(Y)=(A^rYA^r)^{1/q}.$$

In general, $A^p\geqq B^p$ does not always hold for any $p>1$ even if $A\geqq B\geqq 0$. But Theorem 1 asserts that this order holds in this function, that is, hypothesis $A\geqq B\geqq 0$ assures

$$f(A^p)\geqq f(B^p)$$

namely

$$(B^rA^pB^r)^{1/q}\geqq(B^rB^pB^r)^{1/q}=B^{(p+2r)/q}$$

and also

$$g(A^p)\geqq g(B^p)$$

namely

$$A^{(p+2r)/q}=(A^rA^pA^r)^{1/q}\geqq(A^rB^pA^r)^{1/q}$$

hold under the conditions in Theorem 1.

Recently, F. Hansen [3] has given an ingenious and elegant proof to the following inequality by using a unitary dilation of a contraction.

Theorem B. *Let X and Y be bounded linear operators on a Hilbert space. We suppose that $X\geqq 0$ and $\|Y\|\leqq 1$. If f is an operator monotone function defined on* $[0, \infty[$, *then*

$$f(Y^*XY)\geqq Y^*f(X)Y.$$

By using Theorems A and B, we can give a simplified proof of Theorem 1. Proof of Theorem 1 will appear elsewhere together with related counterexamples without conditions in Theorem 1 by Acos 850 computer at the Information Processing Center in Hirosaki University.

Remark 2. Theorem 1 yields the famous Theorem A when we put $r=0$ in Theorem 1. Put $p\geqq 1$ and $(1+2r)q=p+2r$ in Theorem 1, then we have Corollary 1. Also put $p=q$ in Theorem 1, then we have Corollary 2. Also Corollary 2 implies that $A\geqq B\geqq 0$ assures $(BA^pB)^{1/p}\geqq B^{(p+2)/p}$ for each $p\geqq 1$ and this inequality for $p=0$ in matrix case is just an affirmative answer to a conjecture posed by Chan and Kwong [2] Moreover we show more stronger result than this conjecture by using Theorem 1.

Remark 3. We would like to explain the conditions in Theorem 1. For some given $r\geqq 0$, any point (p, q) satisfying the conditions $p\geqq 0$, $q\geqq 1$ and $(1+2r)q\geqq p+2r$ in Theorem 1 lies in the domain surrounded by the oblique lines in Figure and Theorem 1 holds for this point (p, q). Roughly speaking, Theorem 1 would hold for almost all point (p, q) with $p\geqq 0$ and $q\geqq 1$ attending r to infinity.

Also Theorem 1 holds for any point (p, q) satisfying the conditions $q\geqq 1$ and $q\geqq p\geqq 0$ for the restricted $r=0$, this result is just Theorem A.

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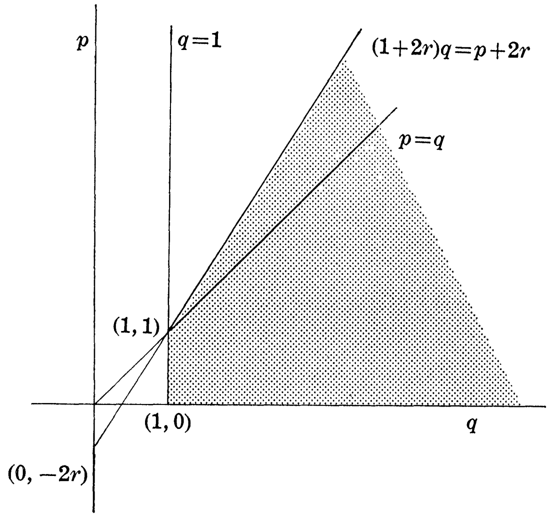

Figure

In [9] the operator equation $THT=K$ had been considered as a useful tool for noncommutative Radon-Nikodym theorem. As an application of Theorem 1, we have Theorem 2 closely related to this operator equation $THT=K$.

Theorem 2. *Let H and K be positive operators and assume that H is nonsingular. If there exists the positive operator T such that $T(H^{1/n}T)^n=K$ for some natural number n, then there exists the positive operator T_1 such that $T_1(H^{1/m}T_1)^m=K$ for any natural number m such that $m\leqq n$. In each case, there is at most one positive solution T and T_1 respectively.*

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