113. On a Certain Distribution on $GL(n)$ and Explicit Formulas

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1. A. Weil [3] constructed a universal distribution Δ on the Weil group. The values of Δ at various test functions give the contributions from the zeros of L-functions which appear in the explicit formulas. In this note, we shall construct a universal distribution Δ_n on $GL(n)$ and prove the explicit formula for automorphic L-functions using Δ_n when $n=2$. For $n>2$, to derive such a result, we must assume certain property of characters of infinite dimensional representations of $GL(n)$ over a local field. This property, formulated as Conjecture, seems to lie slightly beyond our present knowledge of harmonic analysis. The distributions Δ_n have striking formal resemblance to Weil's one. Furthermore they are related to each other so that Δ_m is the "direct image" of Δ_n for $m>n$. This is a pleasant fact since we think that a discovery of new functorial properties related to zeros of zeta functions would be crucial for the proof of the Riemann hypothesis.

2. Let k be a global field of characteristic p. We shall chiefly be concerned with the number field case. For $p>1$, modifications are suggested when necessary. Let $G=GL(n)$ considered as an algebraic group defined over k and let G_A denote the adelization of G. Let T denote the maximal split torus consisting of all diagonal matrices in G. Let $\pi=\otimes\pi_v$ be a cuspidal automorphic representation of G_A and $L(s,\pi)=\prod_v L(s,\pi_v)$ be the L-function attached to π. Then $L(s,\pi)$ is an entire function which satisfies the functional equation

(1) $$L(s,\pi)=\varepsilon(s,\pi)L(1-s,\tilde{\pi}).$$

For $F\in C_c^\infty(\boldsymbol{R}_+^\times)$ and $s\in\boldsymbol{C}$, set

(2) $$\Phi(s)=\int_{\boldsymbol{R}_+^\times}F(x)x^{1/2-s}d^\times x.$$

(If $p>1$, set $\Phi(s)=\sum_{n\in\boldsymbol{Z}}F(q^n)q^{n(1/2-s)}\log q$, where q is the number of elements of the constant field of k.) Then Φ is an entire function of s. It satisfies $\Phi(\sigma+it)=O(|t|^{-N})$, $|t|\to\infty$ for any N uniformly for any fixed vertical strip $A\le\sigma\le B$ if $p=0$. Without losing substantial generality, we assume that π is unitary. Let $A>1/2$, $T'>T$ and R be the rectangle whose vertexes are $1/2\pm A+iT$, $1/2\pm A+iT'$. Let C denote the contour ∂R taken in positive direction. As usual, we consider the integral

(3) $$I(T,T')=\frac{1}{2\pi i}\int_C\Phi(s)d\log L(s,\pi),$$

assuming that no zeros of $L(s,\pi)$ lie on C. It is easy to check the existence of

$$I := \lim_{T\to-\infty, T'\to+\infty} I(T, T')$$

(a weak version of the Riemann-von Mangoldt formula, $N(T)=O(T\log T)$ in usual notation, suffices). By [1], Remark 5.4, $L(s,\pi)$ does not vanish outside of the vertical strip $0<\sigma<1$. Set

$$S(\pi, F)=\sum_{\rho} n(\rho,\pi)\Phi(\rho), \tag{4}$$

where ρ extends over all zeros of $L(s,\pi)$ with the multiplicity $n(\rho,\pi)$. (If $p>1$, we count zeros modulo iP, where $P=2\pi/\log q$.) By the residue calculus, we get $I=S(\pi,F)$. As $\varepsilon(s,\pi)=\text{constant}\times(|f(\pi)||d_k|^n)^s$, using the functional equation, we obtain

$$S(\pi,F)=J-F(1)\log(|f(\pi)||d_k|^n), \tag{5}$$

where

$$J=\frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty}\Phi(1/2+A+it)d\log L(1/2+A+it,\pi)\right.$$
$$\left.-\Phi(1/2-A+it)d\log L(1/2+A-it,\tilde{\pi})\right\}$$

and $|f(\pi)|$ (resp. $|d_k|$) denotes the idele norm of the conductor of π (resp. the differential idele of k). Take A sufficiently large. As

$$\log L(s,\pi)=\sum_v \log L(s,\pi_v)$$

when the Euler product is absolutely convergent, we see easily that J can be divided into local contributions.

$$J=\sum_v J_v,$$
$$J_v=\frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty}\Phi(1/2+A+it)d\log L(1/2+A+it,\pi_v)\right.$$
$$\left.-\Phi(1/2-A+it)d\log L(1/2+A-it,\tilde{\pi}_v)\right\}.$$

(If $p>1$, the integrals for J and J_v should be taken from a to $a+P$ for some $a\in \boldsymbol{R}$).

3. We are going to express J_v using the characters of π_v. Let Δ_v be the absolute value of the discriminant function of G_v. We have $\Delta_v(g)=|\prod_{i<j}(\lambda_i-\lambda_j)^2|_v/|\det g|_v^{n-1}$, where λ_i $(1\le i\le n)$ are eigenvalues of $g\in G_v$. Let χ_{π_v} denote the character of π_v and we set $\tilde{\chi}_{\pi_v}=\chi_{\pi_v}\times\Delta_v^{1/2}$. Hereafter until 5, we shall assume $n=2$. First let v be non-archimedean. Let $\mathcal{O}_v$ denote the maximal compact subring of k_v, q_v the module of k_v and we normalize the multiplicative Haar measure $d^\times\lambda$ on $k_v^\times$ so that $\mathrm{vol}(\mathcal{O}_v^\times)=\log q_v$. For a continuous function f on $k_v^\times$, we set

$$PF_0\int_{k_v^\times}\frac{f(\lambda)}{|\lambda-1|_v}d^\times\lambda=\int_{\mathcal{O}_v^\times}\frac{f(\lambda)-f(1)}{|\lambda-1|_v}d^\times\lambda+\int_{k_v^\times-\mathcal{O}_v^\times}\frac{f(\lambda)}{|\lambda-1|_v}d^\times\lambda,$$

whenever the integrals are meaningful. We have

$$J_v-F(1)\log(|f(\pi_v)|_v) \tag{6}$$
$$=-PF_0\int_{k_v^\times}F(|\lambda|_v)\tilde{\chi}_{\pi_v}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\frac{|\lambda|_v^{1/2}}{|\lambda-1|_v}d^\times\lambda.$$

We note that $\tilde{\chi}_{\pi_v}$ can be considered as a continuous function on T_v.

Now let v be an archimedean place. Define functions on $\boldsymbol{R}_+^\times$ by

$$f_0(x)=\inf(x^{1/2}, x^{-1/2}), \qquad f_1=f_0^{-1}-f_0.$$

If φ is a function on $\boldsymbol{R}_+^\times$ such that $\varphi-cf_1^{-1}$ is integrable on $\boldsymbol{R}_+^\times$, we set

$$PF_0\int_{\boldsymbol{R}_+^\times}\varphi(x)d^\times x=\lim_{t\to+\infty}\left\{\int_{\boldsymbol{R}_+^\times}(1-f_0^{2t}(x))\varphi(x)d^\times x-2c\log t\right\}+2c\log 2\pi.$$

Let $k_v^0=\{x\in k_v^\times \mid |x|_v=1\}$. For a function f on $k_v^\times$, define the function φ on $\boldsymbol{R}_+^\times$ by

$$\varphi(x)=\int_{k_v^0}f(yz)dz \qquad \text{with } |y|_v=x,$$

and set

$$PF_0\int_{k_v^\times}f(\lambda)\,d^\times\lambda=PF_0\int_{\boldsymbol{R}_+^\times}\varphi(x)d^\times x,$$

whenever the right hand side is meaningful (cf. [3], § 3 and § 14). Then we have

$$J_v=-PF_0\int_{k_v^\times}F(|\lambda|_v)\tilde{\chi}_{\pi_v}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\frac{|\lambda|_v^{1/2}}{|\lambda-1|_v}d^\times\lambda, \tag{7}$$

where $\tilde{\chi}_{\pi_v}$ is considered as a continuous function on T_v.

4. For every place v of k, define a distribution D_v on G_v by

$$D_v(f)=-PF_0\int_{k_v^\times}f\left(\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\right)\frac{|\lambda|_v^{1/2}}{|\lambda-1|_v}d^\times\lambda.$$

Let δ_1 be the Dirac distribution on G_A supported on 1. We define a distribution Δ on G_A by

$$\Delta=-\log|d_k|\delta_1-\sum_v D_v,$$

where v extends over all places of k. The function $g_v\to(1/2)\tilde{\chi}_{\pi_v}(g_v)$ is a continuous function on T_v. If $g\in G_A$, its value is 1 or 0 except for finitely many v. Hence we may set

$$\tilde{\chi}_\pi(g)=\prod_v\frac{1}{2}\tilde{\chi}_{\pi_v}(g_v), \qquad g\in T_A$$

and we may consider the pairing with Δ, since Δ is supported on T_A. By (4), (5), (6), (7), we can state the final result as follows.

Theorem. $S(\pi, F)=\Delta(2F(|\det g|)\tilde{\chi}_\pi(g))$.

Remark. Our distribution Δ lacks the term which corresponds to D in Weil's formula. This is simply because we have only considered *cuspidal* automorphic forms. It would be interesting to construct the distribution which corresponds to D using the theory of Eisenstein series.

5. We shall describe the general situation for $GL(n)$, $n>2$. Set $g_\lambda=\operatorname{diag}[\lambda, 1, \cdots, 1]\in G_v$ for $\lambda\in k_v^\times$.

Conjecture. (1) Assume v is non-archimedean and π_v is generic. Then $\tilde{\chi}_{\pi_v}$ is continuous on T_v and $\tilde{\chi}_{\pi_v}(1)=n!$. Furthermore

$$(n-1)!\,J_v=-\int_{k_v^\times-\mathcal{O}_v^\times}F(|\lambda|_v)\tilde{\chi}_{\pi_v}(g_\lambda)\frac{|\lambda|_v^{1/2}}{|\lambda-1|_v}d^\times\lambda,$$

$$(n-1)!\log(|f(\pi_v)|_v)=\int_{\mathcal{O}_v^\times}(\tilde{\chi}_{\pi_v}(g_\lambda)-n!)\frac{1}{|\lambda-1|_v}d^\times\lambda.$$

Thus we have

$$(n-1)!\,[J_v - F(1)\log(|f(\pi_v)|_v)] = -PF_0\int_{k_v^\times} F(|\lambda|_v)\tilde{\chi}_{\pi_v}(g_\lambda)\frac{|\lambda|_v^{1/2}}{|\lambda-1|_v}d^\times\lambda.$$

(2) If v is archimedean and π_v is generic, we have

$$(n-1)!\,J_v = -PF_0\int_{k_v^\times} F(|\lambda|_v)\tilde{\chi}_{\pi_v}(g_\lambda)\frac{|\lambda|_v^{1/2}}{|\lambda-1|_v}d^\times\lambda.$$

Admitting this conjecture, we can express $S(\pi, F)$ as follows. For every place v of k, define a distribution D_v on G_v by

$$D_v(f) = -PF_0\int_{k_v^\times} f(g_\lambda)\frac{|\lambda|_v^{1/2}}{|\lambda-1|_v}d^\times\lambda.$$

Let δ_1 be the Dirac distribution on G_A supported on 1. Set

$$\Delta_n = -\log|d_k|\delta_1 - \sum_v D_v,$$

which is a distribution on G_A. Put

$$\tilde{\chi}_\pi(g) = \prod_v \frac{1}{n!}\tilde{\chi}_{\pi_v}(g_v),$$

which is meaningful at least for $g \in T_A$. Then we have

$$S(\pi, F) = \Delta_n(n\,F(|\det g|)\tilde{\chi}_\pi(g)).$$

We note that Δ_n has a good functorial property. For $m > n$, let $\iota^{m,n}$ denote the standard injection of $GL(n)_A$ into $GL(m)_A$ given by

$$g \longrightarrow \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have $\iota_*^{m,n}\Delta_n = \Delta_m$, where $\iota_*^{m,n}\Delta_n$ denotes the direct image of Δ_n under $\iota^{m,n}$.

References

[1] H. Jacquet and J. A. Shalika: On Euler products and the classification of automorphic representations. I. Amer. J. Math., **103**, 499–558 (1981).

[2] C. J. Moreno: Explicit formulas in the theory of automorphic forms. Springer Lecture Notes, **626**, 73–216 (1976).

[3] A. Weil: Sur les formules explicites de la théorie des nombres. Izv. Math. Nauk, **36**, 3–18 (1972) = [1972] in OEuvres (see also [1952b] in OEuvres).