112. *Gram's Law for the Zeta Zeros and the Eigenvalues of Gaussian Unitary Ensembles*

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§1. Let γ run over the positive imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$. Let $\Gamma(s)$ be the Γ-function and $\vartheta(t)=\mathrm{Im}(\log\Gamma(1/4+(1/2)it))-(1/2)t\log\pi$, where t is a real number. We define g_x by $\vartheta(g_x)=x\pi$ for $x\geqq -1$, where $\vartheta(t)$ is strictly increasing for $t\geqq 7$. Here we are concerned with the following problem.

Problem. *To study the quantity defined by*

$$\lim_{M\to\infty}\frac{1}{M}G_M(k,\alpha)$$

for each integer $k\geqq 0$ and any positive number α, where we put

$$G_M(k,\alpha)=\left|\left\{-1\leqq m\leqq M\,;\left|\left\{\gamma\leqq\vartheta^{-1}(\pi(M+1))\,;\frac{1}{\pi}\vartheta(\gamma)\in[m,m+\alpha)\right\}\right|=k\right\}\right|,$$

γ being counted with multiplicity.

We recall two observations concerning this problem. First Gram observed more than eighty years ago that the zeros of $\zeta(1/2+it)$ appears exactly once in the interval (g_m, g_{m+1}) up to $t\leqq 50$. This phenomenon, which seemed to Gram to continue also for $t>50$, has been called Gram's law although many counter-examples have been observed since Hutchinson (cf. chapters 6, 7 and 8 of [2] for a detailed description of the history). Gram's law implies that for any integer $M\geqq -1$, $G_M(k,1)=M+2$ if $k=1$ and $=0$ if $k\neq 1$. Second, the latest computer calculations by van de Lune, te Riele and Winter [14] tell us that for $M=1500000000$, $(1/M)G_M(0,1)=0.1378\cdots$, $(1/M)G_M(1,1)=0.7261\cdots$, $(1/M)G_M(2,1)=0.1342\cdots$ and $(1/M)G_M(3.1)=0.0018\cdots$ and that $(1/M)G_M(k,1)$ increases for $k=0$, 2 and 3 and decreases for $k=1$ as M becomes larger. We remark here that in both observations, all the non-trivial zeros of $\zeta(s)$ are on the critical line and are simple as far as they have calculated. In this note we shall state some results and conjectures concerning the above problem.

§2. We denote the number of the non-trivial zeros of $\zeta(s)$ in $0<\mathrm{Im}(s)<t$ by $N(t)$ as usual. Since

$$G_M(k,\alpha)=|\{-1\leqq m\leqq M\,;N(g_{m+\alpha})-N(g_m)=k\}|$$

and $N(t)=\pi^{-1}\vartheta(t)+1+S(t)$ for $t\geqq t_0$, the following means

$$\sum_{m\leqq M}(S(g_{m+\alpha})-S(g_m))^j \qquad \text{for any integer } j\geqq 1$$

must give some information on our problem, where we put $S(t)=(1/\pi)arg\,\zeta(1/2+it)$ as usual. The above sum is a discrete version of the integral

$$\int_0^T (S(t+h)-S(t))^j dt,$$

which has been studied by the author in [4]. Extending the author's approach in [5], where we have treated the sum

$$\sum_{m\leq M}\left(S\left(\frac{2\pi\alpha(m+1)}{\log(m+1)}\right)-S\left(\frac{2\pi\alpha m}{\log m}\right)\right)^2,$$

we can prove the following theorem.

Theorem. *Suppose that $M>M_0$, j is an integer $\geqq 1$ and positive α satisfies $\alpha \ll \log M$. We put $\tilde{\alpha}=\mathrm{Max}(1,\alpha)$. Then we have*

$$\sum_{m\leq M}(S(g_{m+\alpha})-S(g_m))^l$$

$$=\begin{cases}\dfrac{(2j)!}{(2\pi)^{2j}j!}M(2\log\tilde{\alpha})^j+O(M(Aj)^j(j^j+(\log\tilde{\alpha})^{j-1/2})) & \text{for } l=2j\\ O(M(Aj)^j(j^j+(\log\tilde{\alpha})^{j-1})) & \text{for } l=2j-1,\end{cases}$$

where A is some positive absolute constant and the estimate is uniform with respect to M, j and α.

As consequences we can derive several results on $G_M(k,\alpha)$ as we have derived from our mean value theorem for $S(t+h)-S(t)$ (cf. [3]). (We may remark here that one gets the same dependence on l in our mean value theorem $\int_0^T (S(t+h)-S(t))^l dt$ if one is a little bit careful at the last stage of its proof (cf. Joyner [7]).) We may state some of them as follows.

Corollary. (i) *Suppose that $\alpha=\alpha(M)$ tends to ∞ as M tends to ∞. Then we have for any real β,*

$$\lim_{M\to\infty}\frac{1}{M}\left|\left\{m\leq M\,;\,\frac{N(g_{m+\alpha})-N(g_m)-\alpha}{\sqrt{2\log\alpha}/2\pi}<\beta\right\}\right|=\int_{-\infty}^{\beta}e^{-x^2}dx.$$

(ii) *Let $\alpha>\alpha_0$ and $M>M_0$. Then there exists some positive constant A such that*

$$\sum_{k\geq\alpha+A\log\alpha\cdot\log\log\alpha}G_M(k,\alpha)\gg M \quad \textit{and}$$

$$\sum_{0\leq k\leq\alpha-A\log\alpha\cdot\log\log\alpha}G_M(k,\alpha)\gg M.$$

(iii) *If $\alpha>\alpha_0$, then for some $k=0,1,2,\cdots$,*

$$\lim_{M\to\infty}\frac{1}{M}G_M(k,\alpha)\neq\frac{e^{-\alpha}\alpha^k}{k!}.$$

(ii) implies, in particular, that $G_M(0,1)\gg M$ and $\sum_{k=2}^{\infty}G_M(k,1)\gg M$. This result is stated in p. 199 of Selberg [12]. (iii) implies that the distribution is not Poisson. The numerical computations quoted above also indicate that the distribution may not be Poisson even in the case $\alpha=1$.

§3. Here we shall state some conjectures concerning our problem. We start with recalling the well known conjecture of Montgomery [9] in the following form.

Montgomery's conjecture. *For $0<\alpha_0<\alpha<\alpha_1<\infty$,*

$$\left|\left\{0<\gamma,\gamma'<T\,;\,0<\gamma'-\gamma\leq\frac{2\pi\alpha}{\log T}\right\}\right|\sim\frac{T}{2\pi}\log T\int_0^{\alpha}\left(1-\left(\frac{\sin\pi u}{\pi u}\right)^2\right)du,$$

where γ and γ' run over the imaginary parts of the zeros of $\zeta(s)$.

Moreover Dyson seems to have remarked that the function which appears in the right hand side of Montgomery's conjecture is the same as the pair correlation of the eigenvalues of Gaussian Unitary Ensembles (cf. [9]). We next recall that Gallagher and Mueller (cf. p. 208 of [6]) have deduced from Montgomery's conjecture that

$$\int_0^T \left(S\left(t+\frac{2\pi\alpha}{\log T}\right)-S(t)\right)^2 dt \sim (\alpha-\alpha^2+o(\alpha^2))T \quad \text{as } T\to\infty \text{ and } \alpha\to 0.$$

We turn our attentions to our problem. To apply Gallagher-Mueller's argument to the present situation seems to give us a more delicate problem. We need to evaluate $|\{m\leqq M\,;\, m=[\vartheta(\gamma)/\pi]=[\vartheta(\gamma')/\pi],\ 0<\gamma<\gamma' \leqq \vartheta^{-1}((M+1)\pi)\}|$. So instead of connecting our problem with Montgomery's conjecture, we proceed to extend Dyson's remark. We propose first the following.

Conjecture 1. *For each integer $k\geqq 0$ and for $0<\alpha<\alpha_0<\infty$,*

$$\lim_{M\to\infty}\frac{1}{M}G_M(k,\alpha)=E(k,\alpha),$$

where $E(k,\alpha)$ will be defined below.

We define $E(k,\alpha)$ as follows. Let λ_j's run over the eigenvalues of the integral operator

$$\lambda f(y)=\int_{-1}^{1}\frac{\sin((y-x)\pi\alpha)}{(y-x)\pi\alpha}f(x)dx,$$

where the eigen-function f is called a spheroidal function. Then we define for $k\geqq 0$ and for $0<\alpha<\alpha_0$,

$$E(k,\alpha)=\prod_j(1-\lambda_j)\sum_{j_1<\cdots<j_k}\frac{\lambda_{j_1}}{1-\lambda_{j_1}}\cdots\frac{\lambda_{j_k}}{1-\lambda_{j_k}}$$

(cf. 2.32 of Mehta-Cloizeaux [8]). We see that $E(k,1)$ can be computed approximately using p. 350 of [8] as follows. $E(0,1)=0.17$, $E(1,1)=0.74$ and $E(2,1)=0.13$. We may compare these with $(1/M)G_M(k,1)$ for $M=1500000000$ quoted in the first section. We may also state a continuous version of Conjecture 1 as follows.

Conjecture 2. *For each integer $k\geqq 0$ and for $0<\alpha<\alpha_0<\infty$,*

$$\lim_{T\to\infty}\frac{1}{T}\left|\left\{t\leqq T\,;\, N\left(t+\frac{2\pi\alpha}{\log T}\right)-N(t)=k\right\}\right|=E(k,\alpha).$$

We state some consequences of the above conjectures as the following corollary.

Corollary. (i) *For each integer $j\geqq 1$ and for $0<\alpha<\alpha_0$,*

$$\sum_{m\leqq M}(N(g_{m+\alpha})-N(g_m))^j \sim M\sum_{k=0}^{\infty}k^jE(k,\alpha) \quad \textit{as } M\to\infty \quad \textit{and}$$

$$\int_0^T\left(N\left(t+\frac{2\pi\alpha}{\log T}\right)-N(t)\right)^j dt \sim T\sum_{k=0}^{\infty}k^jE(k,\alpha) \quad \textit{as } T\to\infty.$$

(ii) $$\sum_{m\leqq M}(S(g_{m+1})-S(g_m))^2\sim M\left(\frac{\pi^2}{18}-\frac{2\pi^4}{675}+\cdots\right) \quad \textit{as } M\to\infty \quad \textit{and}$$

$$\int_0^T\left(S\left(t+\frac{2\pi\alpha}{\log T}\right)-S(t)\right)^2 dt\sim T(\alpha-\alpha^2+O(\alpha^4)) \quad \textit{as } T\to\infty \textit{ and } \alpha\to 0.$$

(iii) *The sequence* $\vartheta(\gamma_n)/\pi$ *is uniformly distributed mod one* (*in the sense of Weyl*), *where* γ_n *is the n-th* γ.

(ii) is a consequence of (i) and we also use p. 350 of [8]. (iii) is a restatement of (i) for $j=1$ combined with the fact that $\sum_{k=0}^{\infty} kE(k,\alpha)=\alpha$ (cf. II-32 of Bohigas and Giannoni [1]).

Generally, we may say that the sequence of real numbers a_n is GUE distributed if $a_n \leqq a_{n+1} \leqq \cdots \to \infty$ as $n \to \infty$ and a_n satisfies the property of Conjecture 1 (or 2) with a suitable function $F(x)$ which satisfies $|\{a_n \leqq x\}| \sim F(x)$ and plays the same role as $\vartheta(t)/\pi$. So the above conjectures claim that γ_n is GUE distributed. It is an interesting problem to find a sequence in number theory which is GUE distributed.

We remark finally that the eigen-values of the Laplace-Beltrami operator on L^2 (the complex upper half plane$/\Gamma$) for any principal congruence subgroup $\Gamma=\Gamma_p$ of level p a prime $\geqq 3$ is not GUE distributed as a consequence of Randol's argument in [11]. However for the modular group Γ it is not clear.

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