108. *Degenerate Self-Adjoint Perturbation in Hilbert Space*

By Takao NAMBU
Department of Mathematics, Faculty of Engineering, Kumamoto University

(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 14, 1987)

§1. **Introduction.** Among many perturbation operators appearing in differential equations, self-adjoint perturbations constitute a special class because of their nice properties. The purpose of this paper is to develop a theory of a self-adjoint perturbation added to an unbounded self-adjoint operator in a Hilbert space. The perturbation in this paper is a degenerate or a finite-dimensional one which has a physical interpretation as a feedback in control systems. The perturbed operator has a positive parameter. It is studied how the minimum eigenvalue of it behaves as the parameter increases.

We begin with the formulation of the problem. Let H be a real Hilbert space with an inner product and a norm which are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Throughout the paper, L will denote an unbounded self-adjoint linear operator with domain $\mathscr{D}(L)$ dense in H. It is assumed that L is positive definite and has compact resolvent. As is well known [2], there is a set of eigenpairs $\{\lambda_i, \phi_{ij}\}$ for L satisfying the following conditions:

(i) $\sigma(L)=\{\lambda_1, \lambda_2, \cdots\}$; $0<\lambda_1<\lambda_2<\cdots<\lambda_i<\cdots\to\infty$;

(ii) $L\phi_{ij}=\lambda_i\phi_{ij}$, $i\geq 1$, $1\leq j\leq m_i(<\infty)$; and

(iii) the set $\{\phi_{ij}\,;\, i\geq 1, 1\leq j\leq m_i\}$ forms a complete orthonormal system in H.

Given a set $\{\psi_1, \cdots, \psi_N\}\subset H$, let us define an operator B as

$$Bx=\sum_{i=1}^{N}\langle x, \psi_i\rangle\psi_i, \qquad x\in H.$$

It is clear that B is self adjoint and nonnegative. Elements ψ_i's are physically interpreted as sensors and actuators in feedback control systems. The operator B is added to L, and the perturbed operator then becomes

$$L+kB=L+k\sum_{i=1}^{N}\langle\cdot, \psi_i\rangle\psi_i, \tag{1}$$

where k indicates a positive parameter. Since B is bounded, $L+kB$ is also a positive-definite self-adjoint operator with domain $\mathscr{D}(L+kB)=\mathscr{D}(L)$, and has compact resolvent. The minimum eigenvalue of $L+kB$ is denoted by $\mu(k)$, and will play an important role since it determines the decay rate of the semigroup $e^{-t(L+kB)}$ generated by the differential equation in H;

$$\frac{dx}{dt}=-(L+kB)x, \quad t>0, \quad x(0)=x_0.$$

It is easy to derive that

$$\begin{aligned}\mu(k)&=\inf_{x\in\mathscr{D}(L),\|x\|=1}\langle(L+kB)x, x\rangle\\&=\inf_{x\in\mathscr{D}(L^{1/2}),\|x\|=1}\{\|L^{1/2}x\|^2+k\langle Bx, x\rangle\}.\end{aligned}\tag{2}$$

In what follows, several properties of $\mu(k)$ will be derived. First of all, it follows from (2) that

$$\mu(k)\leq\inf_{x\in\mathcal{D}(L^{1/2})\cap\{\psi_1,\cdots,\psi_N\}^{\perp},\|x\|=1}\|L^{1/2}x\|^2.$$

The last term is independent of k and thus an upper bound for $\mu(k)$. For brevity, we set

$$\mathcal{K}_0=\{x\in\mathcal{D}(L^{1/2});\|x\|=1\},\quad \text{and}$$

$$\mathcal{K}_1=\{x\in\mathcal{D}(L^{1/2});\|x\|=1,\ x\perp\psi_1,\cdots,\psi_N\}=\{x\in\mathcal{D}(L^{1/2})\cap\operatorname{Ker}B;\|x\|=1\}.$$

Note that the set $\mathcal{K}_1$ is not empty. In fact, we can always find a $y\neq 0$ such that y is orthogonal to $L^{-1/2}\psi_1,\cdots,L^{-1/2}\psi_N$. Thus, $z=L^{-1/2}y\in\mathcal{D}(L^{1/2})$ is orthogonal to $\psi_1,\cdots,\psi_N$, and $z/\|z\|\in\mathcal{K}_1$.

Let $\{P_n\}$ be a family of linear bounded operators in H such that

(3) $\qquad P_n\to 1$ strongly as $n\to\infty$, and Range $P_n\subset\mathcal{D}(L^{1/2})$.

There are several such families. For example, P_n is given as the orthogonal projection operator mapping H onto span $\{\phi_{ij};1\leq i\leq n,1\leq j\leq m_i\}$. Another example is given by the formula $P_n=n(n+L)^{-1}$. Let us introduce an approximation of B by

$$B_n=\sum_{i=1}^{N}\langle\cdot,P_n\psi_i\rangle P_n\psi_i\geq 0.$$

Operator $L+kB_n$ has properties similar to those of $L+kB$. The minimum eigenvalue of $L+kB_n$ is denoted by $\mu_n(k)$.

§2. Results. The following theorem describes the behaviors of $\mu(k)$ and $\mu_n(k)$ and the relationship between them.

Theorem 1. *$\mu(k)$ is absolutely continuous and monotone nondecreasing in k. In fact, only two cases will occur*; (i) *$\mu(k)$ is strictly increasing, or* (ii) *there is a $k_0<\infty$ such that $\mu(k)$ is constant for $k\geq k_0$. The same is true for $\mu_n(k)$.*

If $\psi_1,\cdots,\psi_N\in\mathcal{D}(L^{1/2})$, we have a relationship

(4)
$$\begin{aligned}\mu_\infty&=\lim_{k\to\infty}\mu(k)=\lim_{k\to\infty}\lim_{n\to\infty}\mu_n(k)=\lim_{n\to\infty}\lim_{k\to\infty}\mu_n(k)\\&=\inf_{x\in\mathcal{K}_1}\|L^{1/2}x\|^2=\min_{x\in\mathcal{K}_1}\|L^{1/2}x\|^2.\end{aligned}$$

The above relationship is shown in the following diagram:

$$\begin{array}{ccc}\mu(k) & \xrightarrow{k\to\infty} & \mu_\infty\\ {\scriptstyle n\to\infty}\uparrow & & \uparrow{\scriptstyle n\to\infty}\\ \mu_n(k) & \xrightarrow[k\to\infty]{} & \mu_{n\infty}.\end{array}$$

Remark. The assumption $\psi_1,\cdots,\psi_N\in\mathcal{D}(L^{1/2})$ is not necessary for the relation $\lim_{n\to\infty}\mu_{n\infty}=\min_{x\in\mathcal{K}_1}\|L^{1/2}x\|^2$.

In the following corollary and theorem, P_n is assumed to be the orthogonal projection operator stated earlier.

Corollary 2. *$\mu_{n\infty}$ is the limit of an algebraic problem in the following sense*:

(5)
$$\mu_{n\infty}=\lim_{m\to\infty}\min_{x\in\mathcal{M}_m}\|L^{1/2}x\|^2,$$

where $\mathcal{M}_m=\{x\in P_mH\cap\operatorname{Ker}B_n;\|x\|=1\}$.

Thus, it becomes somewhat simpler to calculate $\mu_\infty=\min_{x\in\mathcal{K}_1}\|L^{1/2}x\|^2$.

Nevertheless, it seems generally difficult to estimate μ_∞. We obtain, however, the following theorem from another viewpoint:

Theorem 3. *Suppose that $N=m_1+\cdots+m_r$ in (1), and that $\langle P_rBP_rx, x\rangle \geq \|P_rx\|^2$, $x \in H$. Then, we can find a suitable $k^*<\infty$ such that*

$$\mu(k^*)\geq\lambda_1+\frac{1}{1+4a^2}(\lambda_{r+1}-\lambda_1)=\tilde{\alpha}, \tag{6}$$

where $a=\sum_{i=1}^N \|P_r\psi_i\|\,\|(1-P_r)\psi_i\|$. Thus, we have an estimate

$$\|e^{-t(L+k^*B)}\|_{\mathcal{L}(H)}\leq e^{-\tilde{\alpha}t}, \qquad t\geq 0. \tag{7}$$

In Theorem 3, it is generally assumed that $\langle P_rBP_rx, x\rangle\geq c\,\|P_rx\|^2$, $c>0$. However, we may take $c=1$ by adjusting k. We remark that $\tilde{\alpha}$ in (6) can become as large as possible if ψ_i's are chosen so that a^2 does not increase faster than λ_{r+1} as $r\to\infty$. The result of Theorem 3 is applied to a class of linear and/or semilinear parabolic differential equations in H in order to stabilize the evolutions of these equations. The proofs of the above theorems and its application will appear elsewhere.

We close this paper by showing an illustrative finite-dimensional example of (4). The calculation of $\mu(k)$ is simple, but tedious. Thus, the proof is omitted.

Example. Let us consider the case where $H=\boldsymbol{R}^3$, $L=\mathrm{diag}\,[\lambda_1, \lambda_2, \lambda_3]$, $0<\lambda_1<\lambda_2<\lambda_3$, $B=\langle\cdot, \psi\rangle\psi$, and $\psi=(a, 0, b)$. Then,

(i) if $b^2(\lambda_2-\lambda_1)\geq a^2(\lambda_3-\lambda_2)$,

$$\mu(k)=\frac{2\{\lambda_1\lambda_3+(a^2\lambda_3+b^2\lambda_1)k\}}{\lambda_1+\lambda_3+(a^2+b^2)k+\{\{\lambda_3-\lambda_1+(b^2-a^2)k\}^2+4a^2b^2k^2\}^{1/2}}$$

for k large enough; and

(ii) if $b^2(\lambda_2-\lambda_1)<a^2(\lambda_3-\lambda_2)$,

$$\mu(k)=\lambda_2 \qquad \text{for } k \text{ large enough.}$$

On the other hand, $\mathcal{K}_1=\{(x, y, z)\in\boldsymbol{R}^3\,;\, ax+bz=0,\ x^2+y^2+z^2=1\}$. Thus, $\min_{x\in\mathcal{K}_1}\|L^{1/2}x\|^2$ is equal to $\min\{(\lambda_1+\lambda_3a^2b^{-2})x^2+\lambda_2y^2\}$ subject to $(1+a^2b^{-2})x^2+y^2=1$. In each case of (i) and (ii), the relation $\mu_\infty=\min_{x\in\mathcal{K}_1}\|L^{1/2}x\|^2$ is now easily examined.

References

[1] E. Hille and R. S. Phillips: Functional Analysis and Semigroups. Amer. Math. Soc. Colloq. Publ., Providence R.I. (1957).

[2] T. Kato: Perturbation Theory for Linear Operators. Springer-Verlag, New York (1976).

[3] T. Nambu: On stabilization of partial differential equations of parabolic type: Boundary observation and feedback. Funkcial. Ekvac., **28**, 267–298 (1985).

[4] I. E. Segal: Non-linear semi-groups. Ann. of Math. (2), **78**, 339–364 (1963).