

104. The 0-minimal Ideal of the Global of a Combinatorial Completely 0-simple Semigroup

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1. Introduction. Let $S = \mathcal{M}^0(I, M, \{1\}; P)$ be a completely 0-simple combinatorial semigroup where $\{1\}$ is the trivial group and $P = (p_{ji})$ is an $M \times I$ -matrix over $\{0, 1\}$ (see [1], [2]). Let $\mathcal{P}(S)$ be the global i. e. the power semigroup of S . As proved in [3], $\mathcal{P}(S)$ has a unique 0-minimal ideal $\mathcal{C}(S)$, and $\mathcal{C}(S) \cong \mathcal{M}^0(\mathcal{P}(I), \mathcal{P}(M), \{1\}; \tilde{P})$ where $\tilde{P} = (\tilde{p}_{BA})$, $B \in \mathcal{P}(M)$, $A \in \mathcal{P}(I)$, $\tilde{p}_{BA} = 1$ if $p_{ji} \neq 0$ for some $j \in B$, $i \in A$; $\tilde{p}_{BA} = 0$ otherwise. According to [4], $\mathcal{P}(S)$ has a unique maximal regular zero-free \mathcal{G} -class $\mathcal{T}(S)$ and $\mathcal{T}^0(S) = \mathcal{T}(S) \cup \{0\}$ is completely 0-simple and $\mathcal{T}^0(S) \cong \mathcal{M}^0(\bar{Q}(I), \bar{Q}(M), \{1\}; \bar{P})$ where $\bar{Q}(I)$ is the set of all elements A of $\mathcal{P}(I)$ satisfying; there is $j \in M$ such that $p_{ji} \neq 0$ for all $i \in A$; $\bar{Q}(M)$ is dually defined, $\bar{P} = (\bar{p}_{BA})$ where $\bar{p}_{BA} = 1$ if $p_{ji} \neq 0$ for all $j \in B$, $i \in A$; $\bar{p}_{BA} = 0$ otherwise. Let S_1 and S_2 be completely 0-simple semigroups, $S_1 = \mathcal{M}^0(I_1, M_1, \{1\}; P_1)$, $S_2 = \mathcal{M}^0(I_2, M_2, \{1\}; P_2)$. The first author has obtained the following:

Lemma 1. [4] *If $\mathcal{T}^0(S_1) \cong \mathcal{T}^0(S_2)$, then $S_1 \cong S_2$.*

In this paper we will use this result to prove that $\mathcal{C}(S_1) \cong \mathcal{C}(S_2)$ implies $S_1 \cong S_2$.

2. Definitions. Let $X = (x_{ji})$ be an $M \times I$ -matrix (i.e. $j \in M, i \in I$) over $\{0, 1\}$. Given X , define $X' = (x'_{ji})$ by $x'_{ji} = 1$ if $x_{ji} = 0$; $x'_{ji} = 0$ if $x_{ji} = 1$. Let $X = (x_{ji})$ and $Y = (y_{ji})$ be $M_1 \times I_1$ - and $M_2 \times I_2$ -matrices over $\{0, 1\}$ respectively. We say X is *equivalent* to Y , denoted by $X \sim Y$, if there are bijections $\sigma: M_1 \rightarrow M_2$ and $\tau: I_1 \rightarrow I_2$ such that $x_{ji} = y_{\sigma(j), \tau(i)}$ for all $j \in M, i \in I$. If every row and every column of X contains 1, then X is called a *sandwich*. Both X and X' are sandwiches if and only if every row and every column of X contains both 0 and 1. Adjoin a new letter 1 to M and I , namely, let $M^1 = M \cup \{1\}$, $1 \notin M$; $I^1 = I \cup \{1\}$, $1 \notin I$. Given an $M \times I$ -matrix $X = (x_{ji})$ over $\{0, 1\}$, we define an $M^1 \times I^1$ -matrix $X^1 = (x^1_{ji})$ over $\{0, 1\}$ as follows: For all $j \in M, i \in I$,

$$x^1_{ji} = x_{ji}, \quad x^1_{j1} = x^1_{1i} = 0, \quad x^1_{11} = 1.$$

Then $(X^1)'$ is always a sandwich whether X is so or not. Let $X = (x_{ji})$, $j \in M, i \in I$, and for each $B \in \mathcal{P}(M)$, $A \in \mathcal{P}(I)$, define a set $X_{BA} = \{x_{ji} : j \in B, i \in A\}$. Given an $M \times I$ -matrix $X = (x_{ji})$ over $\{0, 1\}$, we define a $\mathcal{P}(M) \times \mathcal{P}(I)$ -matrices \tilde{X} and \bar{X} as follows:

$$\begin{aligned} \tilde{X} = (\tilde{x}_{BA}), \quad \tilde{x}_{BA} &= \begin{cases} 1 & \text{if } 1 \in X_{BA} \\ 0 & \text{if } X_{BA} = \{0\}, \end{cases} \\ \bar{X} = (\bar{x}_{BA}), \quad \bar{x}_{BA} &= \begin{cases} 1 & \text{if } X_{BA} = \{1\} \\ 0 & \text{if } 0 \in X_{BA}. \end{cases} \end{aligned}$$

3. Preliminaries. The following lemma is important for this paper.

Lemma 2. Assume X and Y are $M \times I$ -matrices over $\{0, 1\}$. Then the following hold.

- (1) $(X')' = X$.
- (2) If $X \sim Y$, then $X' \sim Y'$.
- (3) $(\bar{X})' = \tilde{X}'$.
- (4) $\tilde{X} \sim \tilde{Y}$ implies $\tilde{X}^1 \sim \tilde{Y}^1$.
- (5) $X^1 \sim Y^1$ implies $X \sim Y$.

Proof. (1) and (2) are obvious.

(3) Immediate from the definition of X' , \bar{X} and \tilde{X} .

(4) $A\mathcal{P}(M_1^1) \times \mathcal{P}(I_1^1)$ -matrix \tilde{X}^1 can be obtained from a $\mathcal{P}(M_1) \times \mathcal{P}(I_1)$ -matrix $\tilde{X} = (\tilde{x}_{BA})$ as follows:

Let $A^1 = A \cup \{1\}$, $B^1 = B \cup \{1\}$ where $A \in \mathcal{P}(I_1)$, $B \in \mathcal{P}(M_1)$. Entries of \tilde{X}^1 are defined by

$$\tilde{x}_{BA}^1 = \tilde{x}_{BA}^1 \mathbf{1} = \tilde{x}_B^1 \mathbf{1}_A = \tilde{x}_{BA}, \quad \tilde{x}_{B^1 A^1}^1 \mathbf{1} = \tilde{x}_{11}^1 = 1.$$

Similarly \tilde{Y}^1 is obtained from a $\mathcal{P}(M_2) \times \mathcal{P}(I_2)$ -matrix \tilde{Y} . If $\tilde{X} \sim \tilde{Y}$ under bijections $\sigma: \mathcal{P}(M_1) \rightarrow \mathcal{P}(M_2)$ and $\tau: \mathcal{P}(I_1) \rightarrow \mathcal{P}(I_2)$, then we define bijections $\sigma^1: \mathcal{P}(M_1^1) \rightarrow \mathcal{P}(M_2^1)$ and $\tau^1: \mathcal{P}(I_1^1) \rightarrow \mathcal{P}(I_2^1)$ as follows: $\sigma^1 B = \sigma B$ if $B \in \mathcal{P}(M_1)$; $\sigma^1 B^1 = (\sigma B)^1$ if $B \in \mathcal{P}(M_1)$; $\sigma^1 \mathbf{1} = \mathbf{1}$; $\tau^1 A = \tau A$ if $A \in \mathcal{P}(I_1)$; $\tau^1 A^1 = (\tau A)^1$ if $A \in \mathcal{P}(I_1)$; $\tau^1 \mathbf{1} = \mathbf{1}$.

It follows that $\tilde{X}^1 \sim \tilde{Y}^1$ under σ^1 and τ^1 .

(5) Assume $X^1 = (x_{ji}^1)$, $Y^1 = (y_{ji}^1)$ and $X^1 \sim Y^1$ under bijections $\sigma^1: M_1^1 \rightarrow M_2^1$ and $\tau^1: I_1^1 \rightarrow I_2^1$. Let $D_1 \times C_1$ be the set of all (j, i) satisfying the following:

$$x_{ji}^1 = 1, x_{jk}^1 = 0 \text{ for all } k \in I_1, k \neq i; x_{li}^1 = 0 \text{ for all } l \in M, l \neq j.$$

Similarly $D_2 \times C_2$ is defined from Y^1 . Then D_1 is mapped to D_2 under σ^1 , and C_1 is mapped to C_2 under τ^1 . However the submatrices $D_1 \times C_1$ -matrix $X_1 = (x_{ji}^1)$ and $D_2 \times C_2$ -matrix $Y_1 = (y_{ji}^1)$ are invertible matrices. For any bijection $\sigma_2: D_1 \rightarrow D_2$ with $\sigma_2(\mathbf{1}) = \mathbf{1}$, there is a bijection $\tau_2: C_1 \rightarrow C_2$ with $\tau_2(\mathbf{1}) = \mathbf{1}$ such that $X_1 \sim Y_1$ under σ_2 and τ_2 . Now define $\bar{\sigma}_2: M_1^1 \rightarrow M_2^1$ and $\bar{\tau}_2: I_1^1 \rightarrow I_2^1$, respectively such that $\bar{\sigma}_2|D_1 = \sigma_2$, $\bar{\sigma}_2|(M_1^1 \setminus D_1) = \sigma^1|(M_1^1 \setminus D_1)$ and $\bar{\tau}_2|C_2 = \tau_2$, $\bar{\tau}_2|(I_1^1 \setminus C_1) = \tau^1|(I_1^1 \setminus C_1)$. Then $X^1 \sim Y^1$ under $\bar{\sigma}_2$ and $\bar{\tau}_2$, but since $\bar{\sigma}_2(\mathbf{1}) = \mathbf{1}$ and $\bar{\tau}_2(\mathbf{1}) = \mathbf{1}$, we have $X \sim Y$ under $\bar{\sigma}_2|M_1$ and $\bar{\tau}_2|I_1$.

4. Main theorem. In the proof of the following, (1), (2), (3), (4), (5) denote (1), (2), (3), (4), (5) in Lemma 2 respectively.

Theorem. If S_1 and S_2 are combinatorial, then

$$\tilde{P}_1 \sim \tilde{P}_2 \text{ implies } P_1 \sim P_2$$

in other words, $\mathcal{C}(S_1) \cong \mathcal{C}(S_2)$ implies $S_1 \cong S_2$.

$$\begin{aligned} \text{Proof.} \quad \tilde{P}_1 \sim \tilde{P}_2 &\xrightarrow{\text{by (4)}} \tilde{P}_1^1 \sim \tilde{P}_2^1 \xrightarrow{\text{by (1), (3)}} ((\tilde{P}_1^1)') \sim ((\tilde{P}_2^1)') \\ &\xrightarrow{\text{by (1), (2)}} (\tilde{P}_1^1)' \sim (\tilde{P}_2^1)' \xrightarrow{\text{by Lemma 1}} (P_1^1)' \sim (P_2^1)' \\ &\xrightarrow{\text{by (1), (2)}} P_1^1 \sim P_2^1 \xrightarrow{\text{by (5)}} P_1 \sim P_2. \end{aligned}$$

Speaking in terms of semigroups, the theorem says $\mathcal{C}(S_1) \cong \mathcal{C}(S_2)$ implies $S_1 \cong S_2$. On the other hand, if $\mathcal{P}(S_1) \cong \mathcal{P}(S_2)$ then $\mathcal{C}(S_1) \cong \mathcal{C}(S_2)$ ([3]). More-

over the first author has obtained in [4] that if $\mathcal{P}(S_1) \cong \mathcal{P}(S_2)$ then $\mathcal{T}^0(S_1) \cong \mathcal{T}^0(S_2)$. Consequently

Corollary. *If S_1 and S_2 are combinatorial completely 0-simple semigroups, then the following are equivalent*

$$\mathcal{P}(S_1) \cong \mathcal{P}(S_2), \quad \mathcal{T}_1^0(S_1) \cong \mathcal{T}_2^0(S_2), \quad \mathcal{C}(S_1) \cong \mathcal{C}(S_2), \quad S_1 \cong S_2.$$

5. Remark. If S_1 and S_2 are finite, the theorem can be directly proved by means of semilattice homomorphisms without using Lemma 1. This has been obtained by the second author and the proof will be reported in [5].

References

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