101. *Sugawara Operators and their Applications to Kac-Kazhdan Conjecture*

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§1. **Introduction.** Let $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be an affine Kac-Moody Lie algebra of type $X_l^{(1)}$ and its triangular decomposition. A $\mathfrak{g}$-module V is called a highest weight module (HWM) with highest weight (HW) $\lambda\in\mathfrak{h}^*$ if V is generated by a vector $v_\lambda\in V$ such that

$$hv_\lambda=\langle\lambda,h\rangle v_\lambda \quad (h\in\mathfrak{h}) \quad \text{and} \quad \mathfrak{n}_+v_\lambda=0.$$

We call v_λ the highest weight vector of V. There exists the unique $\mathfrak{n}_-$-free HWM $M(\lambda)$ with HW λ. We call it the Verma module of $\mathfrak{g}$ with HW λ. There also exists the unique irreducible HWM with HW λ and we denote it by $L(\lambda)$.

For an HWM V and $\mu\in\mathfrak{h}^*$, set $V_\mu=\{v\in V\mid hv=\langle\mu,h\rangle v\ (h\in\mathfrak{h})\}$. Then V is isomorphic to the direct sum of V_μ's and $\dim V_\mu<\infty$ for each $\mu\in\mathfrak{h}^*$. Hence we can define its formal character by

$$\operatorname{ch} V=\sum_{\mu\in\mathfrak{h}^*}(\dim V_\mu)e^\mu.$$

Here e^μ denotes the formal exponential.

The character of the Verma module is given by

$$\operatorname{ch} M(\lambda)=e^\lambda\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{-\dim\mathfrak{g}_\alpha}.$$

where Δ_+ denotes the set of the positive root of $\mathfrak{g}$.

For a dominant integral weight λ, the character of the irreducible HWM $L(\lambda)$ is well known as the celebrated Weyl-Kac character formula. However it is difficult to determine $\operatorname{ch} L(\lambda)$ for general weight λ. V. G. Kac and D. A. Kazhdan [4] proposed a study of the irreducible HWM $L(-\rho)$ and gave a conjecture:

$$\operatorname{ch} L(-\rho)=e^{-\rho}\prod_{\alpha\in\Delta_+^{\mathrm{re}}}(1-e^{-\alpha})^{-1},$$

where ρ is the normalized half sum of the positive roots, and Δ_+^{re} is the set of positive real roots.

We give the affirmative result for this conjecture in a more general situation.

Definition. Let c be the canonical central element of $\mathfrak{g}$ and $g=\langle\rho,c\rangle$ be the dual Coxeter number of $\mathfrak{g}$. For a $\lambda\in\mathfrak{h}^*$ with the level $\langle\lambda,c\rangle=-g$, we say that λ is a *KK-weight* if $\langle\lambda+\rho,\alpha^\vee\rangle\notin \mathbf{Z}_{>0}$ for each real positive coroot $\alpha^\vee$.

Remark that $-\rho$ is a *KK*-weight. Then one of our main results is the following.

Theorem A. *Let* $\mathfrak{g}$ *be an affine Lie algebra of type* $A_l^{(1)}$, $B_l^{(1)}$ *or* $C_l^{(1)}$.

For each KK-weight λ, *we have*

$$\mathrm{ch}\, L(\lambda)=e^{\lambda}\prod_{\alpha\in\Delta_{+}^{\mathrm{re}}}(1-e^{-\alpha})^{-1}.$$

The purpose of this paper is to introduce the higher order analogy of the so-called Sugawara operators, which enables us to prove Theorem A. However we will only describe the outlines of the discussions, and we leave detailed proofs to [1]. (See also Kac [3] for terminology.)

§ 2. **Sugawara operators.** For the proof of Weyl-Kac formula, it is essential that the Chevalley generators of $\mathfrak{g}$ are locally nilpotent on $L(\lambda)$ for any dominant integral weight λ. Unfortunately, they are not locally nilpotent on $L(\lambda)$ for any KK-weight λ. The calculation of Shapovalov's determinant formula [4] for $M(\lambda)$ suggests us to investigate $\mathfrak{g}$-intertwining operators between the Verma modules $M(\lambda)$'s of KK-weights λ's. Theorem A is obtained as a corollary of the complete description of $\mathfrak{g}$-intertwining operators.

It is well known that the Virasoro algebra acts on $M(\lambda)$ and $L(\lambda)$ for any weight λ with $\langle\lambda, c\rangle\neq -g$ through the Sugawara operators (or the Segal operators). On the other hand, they are $\mathfrak{g}$-intertwining operators between $M(\lambda)$'s for any highest weights λ's of level $-g$.

In this section, we give the abstract definition of the Sugawara operators which is a generalization of the usual one.

Definition. Let $\hat{U}(\mathfrak{g}')$ be the completion of the universal enveloping algebra of $\mathfrak{g}'=[\mathfrak{g},\mathfrak{g}]$. If $s\in\hat{U}(\mathfrak{g}')$ satisfies the following two condition, then we call s a (*higher order*) *Sugawara operator* (or a *Segal operator*) of weight $m\delta$.

(1) The element s is in the inverse image of the center of $\hat{U}(\mathfrak{g}')/(c+g)\hat{U}(\mathfrak{g}')$ by the natural epimorphism from $\hat{U}(\mathfrak{g}')$ onto $\hat{U}(\mathfrak{g}')/(c+g)\hat{U}(\mathfrak{g}')$.

(2) $[h,s]=\langle m\delta, h\rangle s \quad (h\in\mathfrak{h})$.

(Here $\delta\in\Delta$ is the positive imaginary root such that $\Delta^{\mathrm{im}}=\boldsymbol{Z}\delta\backslash\{0\}$ is the set of imaginary root of $\mathfrak{g}$.)

Let π_{λ} be the representation of $\hat{U}(\mathfrak{g}')$ on $M(\lambda)$. The following proposition shows the importance of the Sugawara operators for our problem.

Proposition. *Let* $\lambda\in\mathfrak{h}^{*}$ *be of level* $-g$ *and let* s *be a Sugawara operator of weight* $-m\delta$ $(m>0)$. *Then we have the following.*

(1) *The endomorphism* $\pi_{\lambda}(s)$ *commutes with the action of* $\hat{U}(\mathfrak{g}')$.

(2) *The space* $\pi_{\lambda}(s)M(\lambda)$ *is a* $\mathfrak{g}$-*submodule of* $M(\lambda)$ *isomorphic to either* $M(\lambda-m\delta)$ *or* 0.

Definition. A subset $T=\{T^{j}(m)\mid m\in\boldsymbol{Z}_{<0},\ 0\leq j\leq l-1\}$ of $\hat{U}(\mathfrak{g}')$ is called a *fundamental set of the Sugawara operators* if the following conditions hold.

(1) $T^{j}(m)$ is a Sugawara operators of weight $m\delta$.

(2) If $\lambda\in\mathfrak{h}^{*}$ is of level $-g$, then the set $\{\pi_{\lambda}(s)\mid s\in T\}$ generates a commutative subalgebra of $\mathrm{Hom}_{\boldsymbol{C}}(M(\lambda), M(\lambda))$ which is naturally isomorphic to the polynomial ring $\boldsymbol{C}[\boldsymbol{T}]$.

Theorem B. *Let* $\mathfrak{g}$ *be an affine Lie algebra of type* $\mathrm{A}_{l}^{(1)}, \mathrm{B}_{l}^{(1)}$ *or* $\mathrm{C}_{l}^{(1)}$.

Then there exists a fundamental set of the Sugawara operators.

We will construct explicitly the fundamental set of the Sugawara operators for $A_l^{(1)}$ in the next section.

§3. Construction of the Sugawara operators. It is obvious that if s and t are Sugawara operators of weight $m\delta$ and $n\delta$ respectively, then $[s,t]/(c+g)$ is a well-defined element of $\hat{U}(\mathfrak{g}')$ and is a Sugawara operator of weight $(m+n)\delta$. In case of $A_l^{(1)}, B_l^{(1)}$ and $C_l^{(1)}$, there exist Sugawara operators $T^0(n)$ and $T^1(n)$ such that

$$\{T^0(n), T^1(n), T^j(n)=(c+g)^{-j+1}(adT^1(0))^{j-1}(T^1(n))\,|\,2\leq j\leq l-1,\ n<0\}$$

is a fundamental set of the Sugawara operators.

To construct the operators $T^0(n)$ and $T^1(n)$, we will introduce some type of infinite summations in $\hat{U}(\mathfrak{g}')$, which has its origin in physics. Recall that the non-twisted affine Lie algebra of type $X_l^{(1)}$ has the following realization.

$$\mathfrak{g}=\mathring{\mathfrak{g}}\otimes \boldsymbol{C}[t,t^{-1}]\oplus \boldsymbol{C}c\oplus \boldsymbol{C}d,$$
$$[X(m),Y(n)]=[X,Y](m+n)+m\delta_{m+n,0}(X\,|\,Y)c,$$
$$[d,X(m)]=mX(m),\quad [c,\mathfrak{g}]=0\quad (X,Y\in\mathring{\mathfrak{g}},\ m,n\in\boldsymbol{Z}),$$

where $\mathring{\mathfrak{g}}$ is the simple Lie algebra of type X_l, $X(m)=X\otimes t^m$ and $(\,|\,)$ is an invariant bilinear form of $\mathring{\mathfrak{g}}$.

Definition (*the normal product*). Let $S(\mathring{\mathfrak{g}})$ be the symmetric algebra of $\mathring{\mathfrak{g}}$. We define a map $NP: S(\mathring{\mathfrak{g}})\times \boldsymbol{Z}\to\hat{U}(\mathfrak{g}')$ by

$$NP(X_1X_2\cdots X_k\,;n)$$
$$=\lim_{r\to\infty}\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\sum_{j=0}^{k}\sum_{\substack{p_1+\cdots+p_k=n\\ -r\leq p_1,\cdots,p_k\leq r}}\binom{k}{j}y_j(p_1,\cdots,p_k)X_{\sigma1}(p_1)\cdots X_{\sigma k}(p_k)$$
$$(X_1,X_2,\cdots,X_k\in\mathring{\mathfrak{g}},\ n\in\boldsymbol{Z}).$$

Here

$$\binom{k}{j}=\frac{k!}{j!\,(k-j)!}$$

and $y_j(p_1,\cdots,p_k)$ is defined by

$$y_j(p_1,\cdots,p_k)=y_-(p_1)y_-(p_2)\cdots y_-(p_j)y_+(p_{j+1})\cdots y_+(p_k),$$
$$y_-(p)=\begin{cases}1 & (p<0)\\ 0 & (p\geq 0)\end{cases}\quad\text{and}\quad y_+(p)=\begin{cases}0 & (p<0)\\ 1 & (p\geq 0).\end{cases}$$

We call $NP(X_1X_2\cdots X_k\,;n)$ a normal product of $X_1,X_2,\cdots,X_k\in\mathring{\mathfrak{g}}$.

For $A_l^{(1)}$, the operators $T^0(n)$ and $T^1(n)$ are defined by

$$T^0(n)=NP(C_2\,;n)\quad\text{and}\quad T^1(n)=NP(C_3\,;n)$$

where $C_2=\sum_{i,j=1}^{l+1}E_{ij}E_{ji}$, $C_3=\sum_{i,j,k=1}^{l+1}E_{ij}E_{jk}E_{ki}$ and $E_{\mu\nu}$ is the image of the matrix unit $e_{\mu\nu}=[\delta_{i\mu}\delta_{j\nu}]_{ij}$ by the epimorphism

$$\mathfrak{gl}(l+1\,;\boldsymbol{C})\longrightarrow\mathfrak{gl}(l+1\,;\boldsymbol{C})/\boldsymbol{C}\textstyle\sum_i E_{ii}\simeq\mathfrak{sl}(l+1\,;\boldsymbol{C}).$$

Note that C_2 and C_3 are $\mathring{\mathfrak{g}}$-invariant elements of $S(\mathring{\mathfrak{g}})$ and $T^0(n)$'s are nothing but the "usual" Sugawara operators.

The operator $T^j(m)$ has the following commutation relations.

$$(1)\qquad [T^0(m),E_{\mu\nu}(n)]=-2n(c+g)E_{\mu\nu}(m+n),$$

$$(2)\qquad [T^1(m),E_{\mu\nu}(n)]=-3n(c+g)NP\Big(\textstyle\sum_i E_{\mu i}E_{i\nu}-\dfrac{1}{g}\delta_{\mu\nu}C_2\,;m+n\Big),$$

(3) $[d, T^j(m)]=mT^j(m), \quad [c, T^j(m)]=0 \quad (0\leq j\leq l-1,\ \mathring{\mathfrak{g}}=\mathfrak{sl}(l+1, \boldsymbol{C}))$.
We can prove the algebraic independence of $\pi_\lambda(T^j(m))$'s by calculating the leading term of $T^j(m)$'s with respect to some filtrations.

§4. Further results. Let $\boldsymbol{T}$ be a fundamental set of the Sugawara operators. We regard $\boldsymbol{C}[\boldsymbol{T}]$ as a $\mathfrak{g}$-module by $[\mathfrak{g}, \mathfrak{g}]\boldsymbol{C}[\boldsymbol{T}]=0$ and

$$dT^{j_1}(-m_1)\cdots T^{j_k}(-m_k)=-(m_1+\cdots+m_k)T^{j_1}(-m_1)\cdots T^{j_k}(-m_k).$$

Theorem C. *Let $\mathfrak{g}$ be of type $A_l^{(1)}, B_l^{(1)}$ or $C_l^{(1)}$. If $\lambda, \mu\in\mathfrak{h}^*$ are KK-weights, then*

$$\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu))\simeq \boldsymbol{C}[\boldsymbol{T}]_{\lambda-\mu}.$$

Proposition. *Let $\mathfrak{g}$ and λ be as in the previous theorem. Then we have a following exact sequence of the $\mathfrak{g}$-modules.*

$$\cdots\longrightarrow(\overset{2}{\wedge} V)\otimes M(\lambda)\xrightarrow{d_2}V\otimes M(\lambda)\xrightarrow{d_1}M(\lambda)\longrightarrow L(\lambda)\longrightarrow 0.$$

Here $V=\oplus_{t\in \boldsymbol{T}}\boldsymbol{C}t$ and the map d_k $(k>0)$ is defined by:

$$d_k((T^{j_1}(n_1)\wedge\cdots\wedge T^{j_k}(n_k))\otimes v)$$
$$=\sum_{i=1}^{k}(-1)^{i-1}(T^{j_1}(n_1)\wedge\cdots\wedge T^{j_{i-1}}(n_{i-1})\wedge T^{j_{i+1}}(n_{i+1})\wedge\cdots\wedge T^{j_k}(n_k))\otimes T^{j_i}(n_i)v.$$

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