99. An Extension of a Uniform Asymptotic Stability Theorem by Matrosov

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1. Introduction. The purpose of this paper is to extend Matrosov's result [2] and to obtain sufficient conditions for uniform asymptotic stability of solutions of ordinary differential equations.

The uniform asymptotic stability theorems for autonomous or periodic differential equations were established by Barbashin and Krasovski [3]. These results were generalized to nonautonomous systems by Matrosov [2], Hatvani [1], [3], etc.

Let us consider the nonautonomous ordinary differential equation

(1) $$\dot{x}=X(t,x), \qquad (X(t,0)\equiv 0),$$

where $X:\Gamma\to\boldsymbol{R}^n$ is a continuous function, $\Gamma=\{(t,x)\in\boldsymbol{R}^+\times\boldsymbol{R}^n : \|x\|<H\}$ for some $H>0$, $\boldsymbol{R}^+=[0,+\infty)$, and $\|x\|$ is the Euclidean norm of a point $x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$.

Matrosov [2] gave sufficient conditions for uniform asymptotic stability of zero solution of (1) by using a Liapunov function V which has a negative semi-definite time derivative $\dot{V}_{(1)}$ with respect to (1). In Theorem 1.2 of [2], it is assumed that the function X, its partial derivatives $\partial X/\partial t$, $\partial X/\partial x_i$ $(i=1,2,\cdots,n)$, and the partial derivatives of V: $\partial V/\partial t$, $\partial V/\partial x_i$, $\partial^2 V/\partial t\partial x_i$, $\partial^2 V/\partial x_i\partial x_j$ $(i,j=1,2,\cdots,n)$ are continuous and bounded.

In the present paper, we remove these assumptions and assume more general conditions, which include the above assumptions as a special case.

2. Theorems. For $\varepsilon>0$, let $B_\varepsilon=\{x\in\boldsymbol{R}^n : \|x\|<\varepsilon\}$. The closure of a set $E\subset\boldsymbol{R}^n$ is denoted by $\bar{E}$. A function $a(\cdot)$ is said to belong to class $\mathcal{K}$ if $a(\cdot)$ is a continuous, strictly increasing function on $\boldsymbol{R}^+$ into $\boldsymbol{R}^+$ with $a(0)=0$. Let $A:\Gamma\to\boldsymbol{R}$ be a continuous function which satisfies locally a Lipschitz condition with respect to x. The time derivative of A with respect to (1) is defined by

$$\dot{A}_{(1)}(t,x)=\limsup_{h\to 0+}\frac{1}{h}[A(t+h,\ x+hX(t,x))-A(t,x)] \quad ((t,x)\in\Gamma).$$

Theorem 1. *Suppose that there exists a continuous function $V:\Gamma\to\boldsymbol{R}^+$ which satisfies locally a Lipschitz condition with respect to x. For any α_1,α_2 $(0<\alpha_1<\alpha_2<H)$, let $\Lambda(\alpha_1,\alpha_2)=\{x\in\boldsymbol{R}^n : \alpha_1\le\|x\|\le\alpha_2\}$. Suppose that there exists a continuously differentiable function $W:\boldsymbol{R}^+\times\Lambda(\alpha_1,\alpha_2)\to\boldsymbol{R}$ such that for some $a,b\in\mathcal{K}$ the following conditions hold.*

(i) $a(\|x\|)\le V(t,x)\le b(\|x\|)$ *in* Γ.

(ii) *There exists a continuous function* $U:\Gamma\to\boldsymbol{R}^+$ *such that*

$$\dot V_{(1)}(t,x)\leq -U(t,x)\qquad \text{in } \Gamma.$$

(iii) *For any continuous function* $u:\boldsymbol{R}^+\to B_H$, $U(\cdot,u(\cdot))$ *is uniformly continuous in* $\boldsymbol{R}^+$.

(iv) *There exists* $L=L(\alpha_1,\alpha_2)>0$ *such that*

$$|W(t,x)|<L\qquad \text{in } \boldsymbol{R}^+\times\Lambda(\alpha_1,\alpha_2).$$

(v) *There exist* $l=l(\alpha_1,\alpha_2)>0$ *and* $\xi=\xi(\alpha_1,\alpha_2)>0$ *such that for any* $t\geq 0$ *and any* $x\in J_t=\{x\in\Lambda(\alpha_1,\alpha_2): U(t,x)<l\}$,

$$|\dot W_{(1)}(t,x)|>\xi.$$

Then, the zero solution of (1) *is uniformly asymptotically stable.*

Theorem 2. *Let* $H=+\infty$, *that is,* $\Gamma=\boldsymbol{R}^+\times\boldsymbol{R}^n$ *and* $B_H=\boldsymbol{R}^n$. *If in addition to the assumptions in Theorem* 1, *the function* $a(\cdot)$ *in* (i) *satisfies* $a(r)\to+\infty$ *as* $r\to+\infty$, *then the zero solution of* (1) *is globally uniformly asymptotically stable.*

Corollary 1. *Suppose that there exists a continuously differentiable function* $V:\Gamma\to\boldsymbol{R}^+$, *and for any* α_1,α_2 $(0<\alpha_1<\alpha_2<H)$, *there exists a continuously differentiable function* $W:\boldsymbol{R}^+\times\Lambda(\alpha_1,\alpha_2)\to\boldsymbol{R}$ *such that for some* $a,b\in\mathcal{K}$ *the conditions* (i), (iv) *and the following ones hold.*

(ii′) $\dot V_{(1)}(t,x)\leq 0$ *in* Γ.

(iii′) *For any continuous function* $u:\boldsymbol{R}^+\to B_H$, $\dot V_{(1)}(\cdot,u(\cdot))$ *is uniformly continuous in* $\boldsymbol{R}^+$.

(v′) *There exist* $l=l(\alpha_1,\alpha_2)>0$ *and* $\xi=\xi(\alpha_1,\alpha_2)>0$ *such that for any* $t\geq 0$ *and any* $x\in J'_t=\{x\in\Lambda(\alpha_1,\alpha_2):\dot V_{(1)}(t,x)>-l\}$,

$$|\dot W_{(1)}(t,x)|>\xi.$$

Then, the zero solution of (1) *is uniformly asymptotically stable.*

If, in addition, $H=+\infty$ *and the function* $a(\cdot)$ *in* (i) *satisfies* $a(r)\to+\infty$ *as* $r\to+\infty$, *then the zero solution of* (1) *is globally uniformly asymptotically stable.*

Corollary 2. *In Corollary* 1, *if* $V\in C^2$ *and* $X\in C^1$, *then the condition* (iii′) *can be replaced by the following one.*

(iii″) *Let* $A(t,x)\equiv\dot V_{(1)}(t,x)$. *For any continuous function* $u:\boldsymbol{R}^+\to B_H$, $\int_0^t \dot A_{(1)}(s,u(s))ds$ *is uniformly continuous in* $\boldsymbol{R}^+$.

Remark. Corollary 2 includes Theorem 1.2 of [2] as a special case.

3. **Proofs.** *Proof of Theorem* 1. The conditions (i) and (ii) imply that the zero solution of (1) is uniformly stable, that is, for any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that for every $t^*\in\boldsymbol{R}^+$, $x^*\in B_\delta$, any solution $x(\cdot\,;t^*,x^*)$ of (1) and any $t\geq t^*$,

$$\|x(t\,;t^*,x^*)\|<\varepsilon. \tag{2}$$

Let us choose $h>0$ such that $h<H$. It also follows from (i) and (ii) that there exists $\eta>0$ such that for every $t_0\geq 0$, $x_0\in B_\eta$, any solution $x(\cdot\,;t_0,x_0)$ of (1) and any $t\geq t_0$,

$$\|x(t\,;t_0,x_0)\|<h. \tag{3}$$

Let $x(\cdot)\equiv x(\cdot\,;t_0,x_0)$, $\alpha_1=\delta=\delta(\varepsilon)$ and $\alpha_2=h$. Then, there exist constants

$L=L(\varepsilon)>0$, $l=l(\varepsilon)>0$ and $\xi=\xi(\varepsilon)>0$ given in the conditions (iv) and (v). Since (iii) implies that $U(\cdot, x(\cdot))$ is uniformly continuous in $\boldsymbol{R}^+$, there exists $\sigma=\sigma(\varepsilon)>0$ such that for any $t, t'\geq 0$ satisfying $|t-t'|<\sigma$,

$$(4) \qquad |U(t, x(t))-U(t', x(t'))|<\frac{l}{2}.$$

Choose $M\in \boldsymbol{N}$ so that $M>b(\eta)/m$ where $m=\min\{lL/\xi, l\sigma/2\}$. Let $T=2LM/\xi$. Then T depends only on ε.

Now we show that there exists $\bar{t}\in[t_0, t_0+T]$ such that $\|x(\bar{t})\|<\delta$. If it is not true, then by (3), for any $t\in[t_0, t_0+T]$,

$$(5) \qquad \delta\leq\|x(t)\|<h.$$

Suppose that $x(t)\in J_t$ for any $t\in[s_1, s_2]\subset[t_0, t_0+T]$. Then from (iv), (v) and the continuity of $\dot{W}_{(1)}(\cdot, x(\cdot))$,

$$2L>|W(s_2, x(s_2))-W(s_1, x(s_1))|=\left|\int_{s_1}^{s_2}\dot{W}_{(1)}(t, x(t))\,dt\right|$$
$$=\int_{s_1}^{s_2}|\dot{W}_{(1)}(t, x(t))|\,dt\geq\xi(s_2-s_1).$$

Thus, we have

$$(6) \qquad |s_2-s_1|<\frac{2L}{\xi}.$$

Let $[\tau_1, \tau_2]$ be any time interval in $[t_0, t_0+T]$ such that $\tau_2-\tau_1=2L/\xi$. Then, we can consider the following two cases.

Case 1. For any $t\in[\tau_1, \tau_2]$, $U(t, x(t))\geq l/2$.

Case 2. There exists $t_1\in[\tau_1, \tau_2]$ such that $U(t_1, x(t_1))<l/2$.

In Case 1, by (ii), we have

$$(7) \qquad V(\tau_2, x(\tau_2))-V(\tau_1, x(\tau_1))=\int_{\tau_1}^{\tau_2}\dot{V}_{(1)}(t, x(t))dt$$
$$\leq-\int_{\tau_1}^{\tau_2}U(t, x(t))\,dt\leq-\frac{l}{2}(\tau_2-\tau_1)=-\frac{lL}{\xi}.$$

In Case 2, (6) implies that there exists $t_2\in[\tau_1, \tau_2]$ such that $U(t_2, x(t_2))\geq l$. Hence from the continuity of $U(\cdot, x(\cdot))$, there exist $\tau_1', \tau_2'\in[\tau_1, \tau_2]$ such that $U(\tau_1', x(\tau_1'))=l/2$, $U(\tau_2', x(\tau_2'))=l$ and $l/2<U(t, x(t))<l$ for any $t\in(\tau_1', \tau_2')$ if $\tau_1'<\tau_2'$ (or for any $t\in(\tau_2', \tau_1')$ if $\tau_2'<\tau_1'$). If $\tau_1'<\tau_2'$, then by (ii), we obtain

$$(8) \qquad V(\tau_2, x(\tau_2))-V(\tau_1, x(\tau_1))\leq V(\tau_2', x(\tau_2'))-V(\tau_1', x(\tau_1'))$$
$$=\int_{\tau_1'}^{\tau_2'}\dot{V}_{(1)}(t, x(t))dt\leq-\int_{\tau_1'}^{\tau_2'}U(t, x(t))dt$$
$$<-\frac{l}{2}|\tau_2'-\tau_1'|.$$

(If $\tau_2'<\tau_1'$, then we also have the inequality $V(\tau_2, x(\tau_2))-V(\tau_1, x(\tau_1))<-l/2|\tau_2'-\tau_1'|$.) On the other hand, since $|U(\tau_2', x(\tau_2'))-U(\tau_1', x(\tau_1'))|=l/2$, (4) implies that $|\tau_2'-\tau_1'|\geq\sigma$. Thus, (8) yields

$$(9) \qquad V(\tau_2, x(\tau_2))-V(\tau_1, x(\tau_1))<-\frac{l\sigma}{2}.$$

From (7), (9) and the definition of m,

$$(10) \qquad V(\tau_2, x(\tau_2))-V(\tau_1, x(\tau_1))\leq-m.$$

Let $t'_i \in [t_0, t_0+T]$ $(i=0, 1, \cdots, M)$ be $t'_i = t_0 + 2iL/\xi$. Then, from (i) and (10), we have

$$0 \leq V(t'_M, x(t'_M)) = V(t_0, x_0) + \sum_{i=1}^{M} \{V(t'_i, x(t'_i)) - V(t'_{i-1}, x(t'_{i-1}))\}$$
$$\leq b(\eta) - mM.$$

Since M has been chosen so that $M > b(\eta)/m$, this is a contradiction.

Thus, there exists $\bar{t} \in [t_0, t_0+T]$ such that $\|x(\bar{t})\| < \delta$, and from (2), $\|x(t)\| < \varepsilon$ for any $t \geq \bar{t}$. This implies that $\|x(t)\| < \varepsilon$ for all $t \geq t_0 + T$, that is, the origin is uniformly attractive. Therefore the zero solution of (1) is uniformly asymptotically stable.

Proof of Theorem 2. From (i), (ii) and the assumption that $a(r) \to +\infty$ as $r \to +\infty$, all solutions of (1) are uniformly bounded, that is, for any $\eta > 0$, there exists $h = h(\eta) > 0$ such that for every $t_0 \geq 0$, $x_0 \in B_\eta$, any solution $x(\cdot\,; t_0, x_0)$ of (1) and any $t \geq t_0$, $\|x(t; t_0, x_0)\| < h$. From this fact and the proof of Theorem 1, it follows that for any $\varepsilon > 0$, there exists $T = T(\varepsilon, \eta) > 0$ such that $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t \geq t_0 + T$. This implies that the origin is globally uniformly attractive. Therefore the zero solution of (1) is globally uniformly asymptotically stable.

Proof of Corollary 1. In Theorems 1 and 2, let $U(t, x) \equiv -\dot{V}_{(1)}(t, x)$. Then we can derive Corollary 1.

Proof of Corollary 2. (iii$''$) implies that $\dot{V}_{(1)}(\cdot, x(\cdot))$ is uniformly continuous in $\boldsymbol{R}^+$, where $x(\cdot)$ is a solution of (1). Therefore if we use $-\dot{V}_{(1)}(\cdot, x(\cdot))$ instead of $U(\cdot, x(\cdot))$ in the proofs of Theorems 1 and 2, then Corollary 2 can be proved.

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