11. *On a Problem of Yamamoto Concerning Biquadratic Gauss Sums*

By Hiroshi ITO
Department of Mathematics, Faculty of Science,
Nagoya University

(Communicated by Shokichi IYANAGA, M. J. A., Feb. 12, 1987)

Yamamoto [4] observed some relations, which can be stated as (I)~(IV) below, between the biquadratic Gauss sums and the generalized Bernoulli numbers defined for prime numbers $p<4,000$ such that $p\equiv 1 \pmod 4$. He proposed the question whether these relations were always true. In this note, we report counter-examples to (I)~(III) and prove (IV). Some remarks on a problem which still remains will be added.

1. For a prime number $p\equiv 1 \pmod 4$, take positive integers a and b such that $p=a^2+4b^2$ and put $\omega=\omega_p:=a+2bi$. Define the Dirichlet character $\chi=\chi_p$ modulo p by $\chi(m)=(m/\omega)_4$ where $(m/\omega)_4$ is the biquadratic residue symbol in $\boldsymbol{Q}(i)$, the Gauss' number field. Consider the Gauss sum

$$\tau(\chi)=\sum_{m=1}^{p-1}\chi(m)e^{2\pi i m/p}.$$

We write

$$\tau(\chi)=\varepsilon_p\omega^{1/2}p^{1/4} \quad \text{with} \quad 0<\arg(\omega^{1/2})<\pi/4.$$

It is a classical result that $\varepsilon_p^4=1$. On the other hand we put

$$A_p=-\frac{1}{p}\sum_{m=1}^{p-1}\chi(m)m,\quad B_p=-\frac{1}{p}\sum_{m=1}^{(p-1)/2}\chi(m)m,\quad C_p=\sum_{m=1}^{(p-1)/2}\chi(m).$$

Then the assertions (I)~(IV) in [4] (p. 212) can be stated as follows:

(I)	$-\pi/4\le\arg(\varepsilon_p\bar{A}_p)<3\pi/4$	if $p\equiv 5 \pmod 8$,
(II)	$-\pi\le\arg(\varepsilon_p\bar{B}_p)<0$	if $p\equiv 5 \pmod 8$,
(III)	$\mathrm{Im}(\varepsilon_p\bar{C}_p)>0$	if $p\equiv 5 \pmod 8$,
(IV)	$\mathrm{Re}(\varepsilon_p\bar{B}_p)>0$	if $p\equiv 1 \pmod 8$,

where the bar indicates the complex conjugation. Note that $A_p=C_p=0$ if $p\equiv 1 \pmod 8$, when the character χ satisfies $\chi(-1)=1$.

At the time when [4] was published, the calculation of ε_p was very hard. Now we have an elegant expression of ε_p obtained by Matthews [3]. Define $\delta_p\in\{1,-1\}$ by

$$\{(p-1)/2\}!\equiv\delta_p i \pmod{\omega}.$$

Then it follows from [3] that

$$\varepsilon_p=-\delta_p\chi(2i)\left(\frac{b}{a}\right)_2\times\begin{cases} i & \text{if } a\equiv 1 \pmod 4,\\ 1 & \text{if } a\equiv 3 \pmod 4,\end{cases}$$

where $(b/a)_2$ is the (rational) Jacobi symbol. By means of this expression for ε_p, the author examined, with the help of an electronic computer, the assertions (I)~(III) for $p<1,000,000$ and found counter-examples. There are 19,623 prime numbers congruent to 5 modulo 8 up to 1,000,000 and the numbers of counter-examples for (I), (II) and (III) are 282, 106 and 1 respectively. The smallest counter-examples are as follows:

(I)	$p=3821$,	$A_p=-17+21i$,	$\varepsilon_p=-i$,
(II)	$p=5477$,	$B_p=1-14i$,	$\varepsilon_p=i$,
(III)	$p=440093$,	$C_p=355+35i$,	$\varepsilon_p=1$.

The assertion (IV) is true. We will give a proof of it in the next paragraph. (Although Yamamoto [4] says that (I) ~ (IV) were verified up to 4,000, we have found the above counter-example to (I) lying in this range.)

2. As is well-known the values at positive integers of the L-series

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s} \qquad (\mathrm{Re}(s)>1)$$

are expressed in terms of the Gauss sum $\tau(\chi)$ and the generalized Bernoulli numbers, and all the assertions from (I) to (IV) can be translated into assertions about $\arg(L(1,\chi))$ or $\arg(L(2,\chi))$. (Concerning this point the author has benefited from a remark made by Heath-Brown and Patterson in Berndt and Evans [1].) This translation gives a proof of (IV) and a viewpoint for understanding the computational result that the numbers of counter-examples for (I) ~ (III) are small.

If we note the relations

$$B_p=-\frac{1}{2}(1+\chi(2))A_p, \quad C_p=2\left(1-\frac{\overline{\chi(2)}}{2}\right)A_p \quad \text{if } p\equiv 5 \pmod 8$$

and

$$B_p=p^{-2}\left(1-\frac{\overline{\chi(2)}}{4}\right)\sum_{m=1}^{p-1}\chi(m)m^2 \qquad \text{if } p\equiv 1 \pmod 8,$$

it is immediately seen that the assertions from (I) to (IV) are equivalent to the following assertions from (I′) to (IV′) respectively:

(I′)	$-3\pi/4\le\arg(L(1,\chi_p)/\omega_p^{1/2})<\pi/4$	if $p\equiv 5 \pmod 8$,
(II′)	$-\pi/2\le\arg((1-\chi_p(2))L(1,\chi_p)/\omega_p^{1/2})<\pi/2$	if $p\equiv 5 \pmod 8$,
(III′)	$\mathrm{Re}(L(1,\chi'_p)/\omega_p^{1/2})>0$	if $p\equiv 5 \pmod 8$,
(IV′)	$\mathrm{Re}(L(2,\chi'_p)/\omega_p^{1/2})>0$	if $p\equiv 1 \pmod 8$,

where χ'_p is the Dirichlet character modulo $2p$ induced from χ_p. Easy calculation using the Euler product for $L(2,\chi'_p)$ shows

$$-\pi/4<\arg(L(2,\chi'_p))<\pi/4$$

and this proves (IV′) and (IV).

The author calculated, by an electronic computer, $\arg(L(1,\chi'_p))$ for prime numbers $p<1,000,000$ such that $p\equiv 5 \pmod 8$. The calculation shows that $\arg(L(1,\chi'_p))$, considered as an element of $\boldsymbol{R}/2\pi\boldsymbol{Z}$, tends to be close to zero (see Table I). This tendency is clearly related to the tendency of the inequalities in (I) ~ (III), or equivalently in (I′) ~ (III′), to be satisfied. Precise formulation of the distribution of $\arg(L(1,\chi'_p))$ will be possible by the method developed in Elliott [2].

Table I

The numbers $F(n)$ of primes p such that $p\equiv 5 \pmod 8$, $p<1,000,000$ and $(n-1)\pi/100\leq\arg(L(1,\chi_p'))<n\pi/100$ are listed. We have $F(n)=0$ for $35\leq n\leq 169$.

n	$F(n)$	n	$F(n)$
1	682	200	773
2	770	199	748
3	725	198	755
4	699	197	713
5	659	196	738
6	672	195	700
7	661	194	596
8	593	193	596
9	557	192	526
10	480	191	488
11	454	190	455
12	440	189	411
13	405	188	378
14	327	187	362
15	317	186	288
16	255	185	238
17	251	184	209
18	174	183	170
19	152	182	168
20	134	181	136
21	101	180	111
22	83	179	92
23	52	178	61
24	42	177	43
25	26	176	44
26	21	175	20
27	11	174	15
28	14	173	5
29	3	172	9
30	7	171	4
31	0	170	1
32	2		
33	0		
34	1		

References

[1] B. C. Berndt and R. J. Evans: The determination of Gauss sums. Bull. Amer. Math. Soc., **5**, 107–129 (1981).

[2] P. D. T. A. Elliott: On the distribution of the values of quadratic L-series in the half-plane $\sigma>1/2$. Invent. math., **21**, 319–338 (1973).

[3] C. R. Matthews: Gauss sums and elliptic functions: II. The quartic sum. Invent. math., **54**, 23–52 (1979).
[4] K. Yamamoto: On Gaussian sums with biquadratic residue characters. J. reine angew. Math., **219**, 200–213 (1965).