96. *Propagation of Singularities for Microdifferential Equations with Multiple Self-tangential Involutory Characteristics*

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§1. **Introduction.** We study a class of microdifferential equations with multiple involutory characteristics. Explicitly, let M be a real analytic manifold of dimension n with a complex neighborhood X and let $\mathfrak{M}$ be a coherent $\mathcal{E}_X$ module defined in a neighborhood of $\rho_0 \in T^*_MX\backslash M$. (See M. Sato *et al.* [4] and P. Schapira [5] for $\mathcal{E}_X$.) We assume that the characteristic variety of $\mathfrak{M}$ is written in a neighborhood of ρ_0 as

(1) $\quad \mathrm{ch}(\mathfrak{M})=\{\rho \in T^*X\,;\, p_1(\rho)\cdot p_2(\rho)\cdots p_l(\rho)=0\}$

by homogeneous holomorphic functions $p_1, \cdots, p_{l-1}$ and p_l defined in a neighborhood of ρ_0. Here $p_1, \cdots, p_{l-1}$ and p_l satisfy the following conditions (2), (3) and (4).

(2) $\quad p_1, \cdots, p_{l-1}$ and p_l are real valued on T^*_MX.

We set $S_j=\{\rho \in T^*_MX\,;\, p_j(\rho)=0\}$ $(1\leq j\leq l)$ and assume

(3) S_j's are regular (non-radical) non-singular hypersurfaces and $\Sigma=\bigcap_{1\leq j\leq l} S_j$ is a regular involutory submanifold of T^*_MX of codimension d.

(4) S_i and S_j are tangent to each other of order $k_0(\geq 1)$ on Σ in case $i\neq j$. This implies that the jets of S_i and S_j coincide up to order k_0 and that S_i and S_j intersect only on Σ if $i\neq j$.

The above class of equations is studied by N. Dencker [1] in C^∞ case and we study the analytic case under somewhat weaker conditions. The author emphasizes here that we pose no assumption on the multiplicities of the equations and that only the geometry of the characteristic varieties is concerned if we employ the theory of microlocal study of sheaves due to M. Kashiwara and P. Schapira [3]. See also N. Tose [9], [10] and [12] for related results about propagation of singularities for systems with involutory characteristics.

§2. **Notation.** To state the results, we give some prerequisites about 2-microfunctions.

Let Λ be a complexifications of Σ in T^*X. Then $\tilde{\Sigma}$ denotes the union of all bicharacteristic leaves of Λ issued from Σ. M. Kashiwara introduced the sheaf C^2_Σ of 2-microfunctions along Σ on $T^*_\Sigma\tilde{\Sigma}$. By C^2_Σ, we can study the properties of microfunctions on Σ precisely. Actually, we have exact sequences

(5) $\quad 0\longrightarrow C_{\tilde{\Sigma}}|_\Sigma \longrightarrow \mathcal{B}^2_\Sigma \longrightarrow \pi_{\Sigma*}(C^2_\Sigma|_{T^*_\Sigma\tilde{\Sigma}\backslash\Sigma})\longrightarrow 0 \qquad (\pi_\Sigma : T^*_\Sigma\tilde{\Sigma}\backslash\Sigma\longrightarrow\Sigma)$

and

(6) $$0 \longrightarrow \mathcal{C}_M|_\Sigma \longrightarrow \mathcal{B}^2_\Sigma.$$

Here $\mathcal{B}^2_\Sigma = \mathcal{C}^2_\Sigma|_\Sigma$ and $\mathcal{C}_{\tilde\Sigma}$ is the sheaf of microfunctions along $\tilde\Sigma$. Moreover we have a canonical spectral map

(7) $$Sp^2_\Sigma : \pi_\Sigma^{-1}(\mathcal{C}_M|_\Sigma) \longrightarrow \mathcal{C}^2_\Sigma,$$

by which we define the 2-singular spectrum for $u \in \mathcal{C}_M|_\Sigma$ as

(8) $$SS^2_\Sigma(u) = \operatorname{supp}(Sp^2_\Sigma(u)).$$

Refer to M. Kashiwara and Y. Laurent [2] and Y. Laurent [4] for more details about 2-microfunctions.

§3. Statement of the result. By the assumption (4), we see easily that the Hamiltonian vector fields $H_{p_1}, \cdots, H_{p_{l-1}}$ and H_{p_l} are tangential on Σ. The phenomenon on Σ of microfunction solutions to $\mathfrak{M}$ is given by

Theorem 1. *Let u be a section of $\mathcal{H}om_{\mathcal{E}_X}(\mathfrak{M}, \mathcal{C}_M)$ defined in a neighborhood of ρ_0. Then* $\operatorname{supp}(u) \cap \Sigma$ *is invariant under H_{p_1}.*

The proof of the theorem above will be given in §4 by stating the corresponding results concerning propagation of 2-microlocal singularities.

§4. Proof of Theorem 1. By finding a suitable real quantized contact transformation, we may assume from the beginning that

(9) $$p_1(z, \zeta) = \zeta_1.$$

Moreover, we may assume

(10) $$\Sigma = \{(x, \sqrt{-1}\xi \cdot dx) \in T^*_M X\,;\, \xi_1 = \cdots = \xi_d = 0\}.$$

Here we take a coordinate of $\sqrt{-1}T^*\boldsymbol{R}^n$ [resp. $T^*\boldsymbol{C}^n$] as $(x, \sqrt{-1}\xi \cdot dx)$ [resp. $(z, \zeta \cdot dz)$] with $x, \xi \in \boldsymbol{R}^n$ [resp. $z, \zeta \in \boldsymbol{C}^n$] and set $\zeta' = (\zeta_1, \cdots, \zeta_d)$. Since S_j is tangent to S_1, we can rewrite p_j as

(11) $$p_j = \zeta_1 + A(z, \tilde\zeta).$$

Here A_j is homogeneous of order 1 and $\tilde\zeta = (\zeta_2, \cdots, \zeta_n)$. By the assumption (4), A_j's are written as

(12) $$A_j(z, \tilde\zeta) = \sum_{|\alpha| = k_0+1} A^\alpha_j(z, \tilde\zeta) \cdot \tilde\zeta'^\alpha$$

with $\tilde\zeta' = (\zeta_2, \cdots, \zeta_d)$. We put $N = \boldsymbol{C}^d_{z'} \times \boldsymbol{R}^{n-d}_{x''}$ in $X = \boldsymbol{C}^n$ with $z' = (z_1, \cdots, z_d)$ and $x'' = (x_{d+1}, \cdots, x_n)$. Then we have

(13) $$\tilde\Sigma \xrightarrow{\sim} T^*_N X$$

and $\mathcal{C}_{\tilde\Sigma}$ is nothing but the sheaf of microfunctions with holomorphic parameters z':

(14) $$\mathcal{C}_{\tilde\Sigma} = \mu_N(\mathcal{O}_X)[n-d].$$

Here $\mu_N(\cdot)$ is the functor of Sato's microlocalization defined in [3]. We take a coordinate of $\tilde\Sigma$ as $(z', x''\,;\, \sqrt{-1}\xi'' dx'')$ with $\xi'' = (\xi_{d+1}, \cdots, \xi_n)$ and that of $T^*_\Sigma\tilde\Sigma$ as $(z', x''\,;\, \sqrt{-1}\xi''\,;\, z'^* dz' + \sqrt{-1}x'^* dx + \sqrt{-1}\xi''^* d\xi'')$ with $z'^* = (z_1^*, \cdots, z_d^*) \in \boldsymbol{C}^d$ and $x''^* = (x^*_{d+1}, \cdots, x^*_n)$, $\xi''^* = (\xi^*_{d+1}, \cdots, \xi^*_n) \in \boldsymbol{R}^{n-d}$.

Moreover, in the case above, we have

(15) $$\mathcal{C}^2_\Sigma = \mu_\Sigma(\mathcal{C}_{\tilde\Sigma})[d] \xrightarrow{\sim} \mu\operatorname{Hom}(\boldsymbol{Z}_\Sigma, \mathcal{C}_{\tilde\Sigma})[d]$$

and

(16) $$\boldsymbol{R}\mathcal{H}om_{\pi^{-1}(\mathcal{E}_X|_\Sigma)}(\pi^{-1}(\mathfrak{M}|_\Sigma), \mathcal{C}^2_\Sigma) \xrightarrow{\sim} \mu\operatorname{Hom}(\boldsymbol{Z}_\Sigma, \mathcal{F})[d] \qquad (\pi : T^*_\Sigma\tilde\Sigma \longrightarrow \Sigma)$$

where $\mathcal{F} = \boldsymbol{R}\mathcal{H}om_{\mathcal{E}_X}(\mathfrak{M}, \mathcal{C}_{\tilde\Sigma})$. (See [3] for the definition of the bifunctor

$\mu\mathrm{Hom}(\cdot,\cdot)$.) We can calculate the microsupport of $\mathcal{F}$ by Theorem 10.5.1. of Kashiwara-Schapira [3] as

$$(17)\qquad SS(\mathcal{F})\subset C_{\tilde{\Sigma}}(ch(\mathfrak{M}))\subset\{(z', x''\,;\sqrt{-1}\xi''\,;z'^*dz'+\sqrt{-1}x'^*dx' +\sqrt{-1}\xi''^*d\xi'')\,;\,z_1^*=0\}.$$

Thus we have

$$(18)\qquad SS(\mathcal{F})\subset\{\mathrm{Re}\,z_1^*=0\}\quad\text{and}\quad SS(\boldsymbol{Z}_\Sigma)=T^*_{\Sigma}\tilde{\Sigma}\subset\{\mathrm{Re}\,z_1^*=0\}.$$

By (16) and (18), we can apply Theorem 5 of N. Tose [11] and show that for any section u of $H^j(\boldsymbol{R}\,\mathcal{H}om_{\pi^{-1}(\mathcal{E}_X|\Sigma)}(\pi^{-1}(\mathfrak{M}|_\Sigma), \mathcal{C}^2_{\tilde{\Sigma}}))$, $\mathrm{supp}\,(u)$ is invariant under $\partial/\partial x_1$. This implies the assertion of Theorem 1. *(q.e.d.)*

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