95. *Expansive Homeomorphisms of Compact Surfaces are Pseudo-Anosov*

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Let (X, d) be a compact metric space and $f: X \to X$ be a homeomorphism. We recall that f is *expansive* (with expansive constant $c>0$) if to each pair (x, y) of distinct points in X there is an integer $n \in Z$ such that $d(f^n(x), f^n(y))>c$. All the examples of expansive homeomorphisms on compact surfaces known so far are pseudo-Anosov diffeomorphisms which are introduced by W. Thurston [4] (cf. [1], [2]). The problem of whether new expansive homeomorphisms exist on compact surfaces is important in topological dynamics. The purpose of this paper is to announce the following result.

Theorem 1. *Every expansive homeomorphism of a compact surface is pseudo-Anosov.*

If this theorem was established, then by using Euler-Poincaré's formula and Kneser's Theorem, we can give a partial answer for the problem of existence of expansive homeomorphisms as follows (cf. [3], [5]).

Theorem 2. *There are no expansive homeomorphisms on the* 2*-sphere, the projective plane and the Klein bottle.*

A homeomorphism f of a compact surface M is *pseudo-Anosov* if there is a pair $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$ of transverse measured foliations with (the same) singularities such that $f(\mathcal{F}^s, \mu^s)=(\mathcal{F}^s, \lambda^{-1}\mu^s)$ and $f(\mathcal{F}^u, \mu^u)=(\mathcal{F}^u, \lambda\mu^u)$ where $\lambda>1$. This means that f preserves the two singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$; it contracts the leaves of $\mathcal{F}^s$ by λ^{-1} and it expands the leaves of $\mathcal{F}^u$ by λ.

Let $x \in X$ and define the *stable* and *unstable sets* $W^s(x)$, $W^u(x)$ as

$$W^s(x)=\{y \in X : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\},$$
$$W^u(x)=\{y \in X : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\}$$

and put

$$\mathcal{F}^\sigma(X, f)=\{W^\sigma(x) : x \in X\} \qquad (\sigma=s, u).$$

Then $\mathcal{F}^\sigma(X, f)$ is a decomposition of X and preserved under f. If X is a compact surface and f is pseudo-Anosov, then it is easily checked that $\mathcal{F}^\sigma = \mathcal{F}^\sigma(X, f)$ $(\sigma=s, u)$.

In order to obtain Theorem 1 we must prove the following

Proposition A. *Let $f: M \to M$ be an expansive homeomorphism. Then the families $\mathcal{F}^\sigma(M, f)$ $(\sigma=s, u)$ have the following properties;*

(1) *$\mathcal{F}^\sigma(M, f)$ is a singular foliation on M,*

(2) *every leaf $W^\sigma(x) \in \mathcal{F}^\sigma(M, f)$ is homeomorphic to* $L_p=\{z \in C : \mathrm{Im}\,(z^{p/2})$

$=0\}$ *for some* $p\geq 2$,

(3) $\mathcal{F}^\sigma(M, f)$ *is minimal,*

(4) $\mathcal{F}^s(M, f)$ *is transverse to* $\mathcal{F}^u(M, f)$.

If Proposition A holds, then the transverse invariant measures μ^σ for $\mathcal{F}^\sigma(M, f)$ $(\sigma=s, u)$ and the stretching factor λ of f are obtained by using the idea in A. J. Casson [1].

For $x\in X$ and $\mathcal{E}>0$, define the *local stable set* $W^s_{\mathcal{E}}(x)$ and the *local unstable set* $W^u_{\mathcal{E}}(x)$ by

$$W^s_{\mathcal{E}}(x)=\{y\in X: d(f^n(x), f^n(y))\leq\mathcal{E},\ n\geq 0\},$$
$$W^u_{\mathcal{E}}(x)=\{y\in X: d(f^n(x), f^n(y))\leq\mathcal{E},\ n\leq 0\}$$

and denote by $C^\sigma_{\mathcal{E}}(x)$ the connected component of x in $W^\sigma_{\mathcal{E}}(x)$ $(\sigma=s, u)$. The following proposition will play a important role in the proof of Proposition A.

Proposition B. *Let* $f: X\to X$ *be an expansive homeomorphism. If* X *is non-trivial, connected and locally connected, then for every* $\mathcal{E}>0$ *there is* $\delta>0$ *such that for all* $x\in X$

$$S_\delta(x)\cap C^\sigma_{\mathcal{E}}(x)\neq\phi \qquad (\sigma=s, u)$$

where $S_\delta(x)=\{y\in X: d(x, y)=\delta\}$.

The details of this paper will appear elsewhere.

References

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