94. Necessary and Sufficient Conditions for the Convergence of Formal Solutions

By Masafumi YOSHINO
Department of Mathematics, Tokyo Metropolitan University

(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1987)

§ 1. **Introduction.** In this paper we shall give a necessary and sufficient condition for the convergence of formal solutions of certain type of analytic equations of general independent variables. The result here is an extension of that of [3] to equations of general independent variables, which coincides with the result of [3] in the case of two independent variables.

§ 2. **Statement of results.** Let $x=(x_1, \cdots, x_d)$ $(d\geq 2)$ be the variable in $\boldsymbol{C}^d$. For $\eta\in\boldsymbol{R}^d$ and a multi-index $\alpha=(\alpha_1, \cdots, \alpha_d)\in\boldsymbol{N}^d$, $\boldsymbol{N}=\{0,1,2,\cdots\}$, we set $\eta^\alpha=\eta_1^{\alpha_1}\cdots\eta_d^{\alpha_d}$ and $(x\cdot\partial)^\alpha=(x_1\partial_1)^{\alpha_1}\cdots(x_d\partial_d)^{\alpha_d}$, where $\partial=(\partial_1, \cdots, \partial_d)$ and $\partial_j=\partial/\partial x_j$ $(j=1, \cdots, d)$. Let $m\geq 1$ be an integer and let $\omega\in\boldsymbol{C}^d$. Then we are concerned with the convergence of all formal solutions of the form $u(x)=x^\omega\sum_{\eta\in\boldsymbol{N}^d}u_\eta x^\eta/\eta!$ of the equation

(2.1) $$P(x;\partial)u\equiv\sum_{|\alpha|\leq m}a_\alpha(x)\partial^\alpha u(x)=f(x)x^\omega$$

where $a_\alpha(x)$ is analytic at the origin and $f(x)$ is a given analytic function. We say that a formal solution $u=x^\omega\sum_{\eta\in\boldsymbol{N}^d}u_\eta x^\eta/\eta!$ converges if the sum $\sum_{\eta\in\boldsymbol{N}^d}u_\eta x^\eta/\eta!$ converges and represents an analytic function in x. Let us expand $a_\alpha(x)$ into the power of x, $a_\alpha(x)=\sum_\gamma a_{\alpha,\gamma}x^\gamma/\gamma!$, and let us define

(2.2) $$M_P=\{\gamma-\alpha\in\boldsymbol{Z}^d;\ a_{\alpha,\gamma}\neq 0 \text{ for some } \alpha \text{ and } \gamma\}.$$

Then we assume

(A.1) $M_P\subset\{\eta\in\boldsymbol{R}^d;\ \eta_1+\cdots+\eta_d\geq 0\}$ and $M_P\cap\{\eta\in\boldsymbol{R}^d;\ \eta_1+\cdots+\eta_d=0\}$ is contained in some proper cone with apex at the origin.

We define the set Γ_0 by $\Gamma_0=$Convex hull of $\{t\theta\in\boldsymbol{R}^d;\ t\geq 0,\ \theta\in M_P\}$.

We set $p(\eta)=\sum_{|\alpha|\leq m}a_{\alpha,\alpha}x^\alpha\partial^\alpha/\alpha!$, and we denote by $p_m(\eta)$ the m-th homogeneous part of $p(\eta)$. For $\xi\in\boldsymbol{R}^d$, $|\xi|=1$, we set $\Gamma(\xi;\varepsilon)=\{\eta\in\boldsymbol{R}^d;\ |\eta/|\eta|-\xi|<\varepsilon\}$. Then we define the quantity $\sigma_{\xi,\varepsilon}$ by

(2.3) $$\sigma_{\xi,\varepsilon}=\sup\{c\in\boldsymbol{R};\ \liminf_{|\eta|\to\infty,\eta\in\Gamma(\xi,\varepsilon)\cap\boldsymbol{Z}^d}|\eta|^{-c}|p(\eta+\omega)|>0\},$$

where if $\liminf|\eta|^{-c}|p(\eta+\omega)|=0$ for every $c\in\boldsymbol{R}$, we put $\sigma_{\xi,\varepsilon}=-\infty$. Note that $\sigma_{\xi,\varepsilon}\leq m$, since $p(\eta+\omega)$ is of degree m. Since $\sigma_{\xi,\varepsilon}$ increases as ε tends to zero, we set $\sigma_\xi\equiv\lim_{\varepsilon\downarrow 0}\sigma_{\xi,\varepsilon}$. For the fundamental property of σ_ξ we refer to [3].

We define a differential operator $Q(x;\partial)\equiv\sum_{|\beta|\leq m_0}b_\beta(x)\partial^\beta$ by

$$Q(x;\partial)=P(x,\partial)-\sum_{|\alpha|\leq m}a_{\alpha,\alpha}x^\alpha/\alpha!,$$

where $m_0\leq m$.

Let us take θ, $|\theta|=1$ such that $p_m(\theta)\neq 0$. We write $\eta=\zeta_1\theta+\zeta'$, $\zeta'=$

$(\zeta_2, \cdots, \zeta_d)$ and factorize

(2.4) $$t^m p(t^{-1}\eta) = t^m p(t^{-1}\zeta_1\theta + t^{-1}\zeta') = p_m(\theta) \prod_{j=1}^{j_0} (\zeta_1 - \lambda_j(\zeta'; t))^{m_j}$$

where $m_j \geq 1$ and $j_0 \geq 1$.

Let $\xi \in \boldsymbol{R}^d$. Then we say that ξ is a smooth characteristic point of $p(\eta)$ if there exists a factorization (2.4) such that for every λ_j satisfying $\xi_1 - \lambda_j(\xi', 0) = 0$ there exist a complex neighborhood $V(\xi)$ of ξ' and a neighborhood V_0 of $t=0$ such that the function $\lambda_j(\zeta' + s\omega; t)$ is differentiable at $s=0$ uniformly in ζ', ω and t on $V(\xi') \times \Gamma_0 \times V_0$, that is

(2.5) $\lim_{s\to 0} s^{-1}\{\lambda_j(\zeta' + s\omega; t) - \lambda_j(\zeta'; t)\}$ exists uniformly on $V(\xi') \times \Gamma_0 \times V_0$.

If $\lambda_j \in C^1$ in ζ' and t then every ξ such that $\xi_1 - \lambda_j(\xi'; 0) = 0$ is a smooth characteristic point. Especially if $\nabla p_m(\xi) \neq 0$ on $p_m(\xi) = 0$ then ξ is a smooth characteristic point.

Let $\xi, \theta \in \boldsymbol{R}^d$, $s \in \boldsymbol{R}$, and let $L_\xi(\theta)$ be the localization of $p_m(\eta)$ at ξ defined by

(2.6) $p_m(\xi + s\theta) = L_\xi(\theta)s^q + 0(s^{q+1}), \quad L_\xi(\theta) \neq 0, \quad q = q(\xi) \geq 0$ integer.

We assume

(A.2) Either $p_m(\xi) \neq 0$ or $\sigma_\xi > m_0$ holds for each $\xi \in \Gamma_0$ and non-smooth characteristic points ξ of $p(\eta)$, and $L_\xi(\theta) \neq 0$ for every $\theta \in \Gamma_0$, $|\theta| = 1$ and smooth characteristic points ξ.

Then we have

Theorem 2.1. *Assume* (A.1) *and* (A.2). *Then all formal solutions of* (2.1) *converge if and only if the following exponential condition is satisfied*

(2.8) $$\liminf_{|\eta|\to\infty, \eta \in N^d} |p(\eta + \omega)|^{1/|\eta|} > 0.$$

In case $d=2$, we can easily show that all characteristic points are smooth in the above sence. Moreover we can show that (A.2) is equivalent to

(A.2)′ Either $p_m(\xi) \neq 0$ or $\sigma_\xi > m_0$ holds for every $\xi \in \Gamma_0$.

Then we have

Corollary 2.2. *Suppose that* $d=2$ *and that the conditions* (A.1) *and* (A.2)′ *are satisfied. Then we have the same assertion as in* Theorem 2.1.

Remark. Corollary 2.2 is proved in [3]. In [(b), Remarks 2.2; 3] a generalization of Corollary 2.2 to equations of general independent variables is also given. But it does not contain Corollary 2.2 because the smoothness of characteristic roots was assumed. Theorem 2.1 improves this point so that it contains Corollary 2.2.

References

[1] Bengel, G. and Gérard, R.: Formal and convergent solutions of singular partial differential equations. Manuscr. Math., **38**, 343–373 (1982).

[2] Kashiwara, M., Kawai, T., and Sjöstrand, J.: On a class of linear partial differential equations whose formal solutions always converge. Ark. für Math., **17**, 83–91 (1979).

[3] Yoshino, M.: The diophantine nature for the convergence of formal solutions. Tôhoku J. Math., **38**(4), 625–641 (1986).