93. *Differential Invariants of Superwebs*

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(Communicated by Kunihiko KODAIRA, M. J. A., Oct. 12, 1987)

§1. **Introduction.** Let M be a superdomain of $2|2$-dimensional super Euclidean space $E^{2|2}$. We consider on M a natural analogy of web (a system of 3-families of curves on a surface), which will be called a superweb.

Blaschke and Dubourdieu ([2]) solved the local equivalence problem of webs by constructing natural torsion free connections. This result implies that all the differential invariants of a web are generated by the curvature.

In this note we shall solve the local equivalence problem of superwebs. We also clarify the relations between superwebs and webs. Details will be published elsewhere.

The author would like to express his gratitude to Prof. T. Tsujishita for his valuable suggestions.

§2. **Distributions on superdomains.** For the sake of brevity, we adopt the Batchelor's formalism of supermanifolds ([1]). In this paragraph, we recall requisites for the subsequent arguments.

A real Grassmann algebra Λ is fixed as the coefficient ring of the theory. It is assumed that the number of odd generators of Λ is sufficiently large (by converting Λ, if needed. cf. [1]). The *$m|n$-dimensional super Euclidean space* $E^{m|n}$ is the direct sum of m copies of Λ_0 and n copies of Λ_1 with the coarse topology. A coarse open set M of $E^{m|n}$ is called a *superdomain*.

A *distribution* $\mathcal{D}$ on M of codimension $r|s$ is locally given by Pfaffian equations $\varphi^1=\cdots=\varphi^r=\psi^1=\cdots=\psi^s=0$, which will be called a *system of local equations* for $\mathcal{D}$. Here φ^i's are even and ψ^j's are odd 1-forms that are independent over the superalgebra of the supersmooth functions. We call $\mathcal{D}$ a *foliation* if it is completely integrable.

§3. **Superwebs.** Let M be a superdomain of dimension $2|2$ and $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ be foliations on M of codimension $1|1$. Let $\varphi_i=\psi_i=0$ be a system of local equations for $\mathcal{D}_i$, $i=1, 2, 3$. We call a triple $\mathcal{W}=\{\mathcal{D}_i\}$ a superweb on M when φ_i, φ_j, ψ_i, ψ_j form a coframe field if $i\neq j$. Two superwebs $\{\mathcal{D}_i\}$ and $\{\mathcal{D}'_i\}$ are called *equivalent* if there is a superdiffeomorphism f such that $f_*(\mathcal{D}_i)=\mathcal{D}'_i$, $i=1, 2, 3$. Such a superdiffeomorphism is called an *equivalence* of the superwebs.

§4. ***GL*(1|1, *Λ*)-structures.** For a superweb $\mathcal{W}=\{\mathcal{D}_i\}$, we can choose, by normalizing, systems of local equations for $\mathcal{D}_i$'s satisfying $\varphi_3=\varphi_1+\varphi_2$ and $\psi_3=\psi_1+\psi_2$. Such systems will be called *normal.* If $\varphi_i=\psi_i=0$ and

$\varphi_i' = \psi_i' = 0$ are two normal systems, then $\varphi_i' = \varphi_i \cdot \alpha + \psi_i \cdot \gamma$ and $\psi_i' = \varphi_i \cdot \beta + \psi_i \cdot \delta$, $i=1, 2, 3$, where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a supersmooth function with values in the *super Lie group* $GL(1|1, \Lambda)$. Thus each superweb $\mathcal{W}$ induces a canonical $GL(1|1, \Lambda)$-*structure*, that is, a $GL(1|1, \Lambda)$-reduction of the principal bundle associated to the tangent bundle of the superdomain M. This induced $GL(1|1, \Lambda)$-structure will be denoted by $P_{\mathcal{W}}$.

It can be easily shown that two superwebs are equivalent if and only if the associated $GL(1|1, \Lambda)$-structures are equivalent. Hence the equivalence problem of superwebs is reduced to that of $GL(1|1, \Lambda)$-structures.

§5. Main results. Consider now general $GL(1|1, \Lambda)$-structures on M. Each $GL(1|1, \Lambda)$-structure on M corresponds, similarly to the manner in §4, to a triple of distributions, which may not be integrable, on M of codimension $1|1$ that are in general position. Such a triple is called an *almost superweb*.

Let $P_{\mathcal{W}}$ be the associated $GL(1|1, \Lambda)$-structure to an almost superweb $\mathcal{W} = \{\mathcal{D}_i\}$. By a method similar to the general theory of G-structures on C^∞-manifold (cf. [4]), we obtain the following theorem.

Theorem 1. *There is a unique supersmooth connection $\Gamma_{\mathcal{W}}$ in $P_{\mathcal{W}}$ such that the torsion $T(X_1, X_2)$ vanishes if X_1 and X_2 are arbitrary vector fields tangent to $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.*

Among almost superwebs, superwebs are characterized as follows.

Theorem 2. *An almost superweb $\mathcal{W}$ is a superweb if and only if $T(X, Y)$ is tangent to $\mathcal{D}_i$ when X and Y are tangent to $\mathcal{D}_i$, $i=1, 2$.*

We remark that the torsions for superwebs, in contrast to webs, are not necessarily zero in general (cf. Remark 2).

§6. Local expressions. In this paragraph, we express the connection associated to a superweb in terms of local coordinates.

Let $\mathcal{W} = \{\mathcal{D}_i\}$ be a superweb. Since each $\mathcal{D}_i$ is completely integrable, there are supersmooth functions x^i and ϑ^i such that $dx^i = d\vartheta^i = 0$ gives a system of local equations for $\mathcal{D}_i$, $i=1, 2, 3$, where x^i's and ϑ^i's are even and odd, respectively. Moreover, since $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are in general position, x^i, x^j, ϑ^i, ϑ^j form a local coordinate system if $i \neq j$. Especially, if we adopt x^1, x^2, ϑ^1, ϑ^2 as a local coordinate system, then dx^3 and $d\vartheta^3$ are expressed as

$$dx^3 = dx^1 \cdot \frac{\vec{\partial} x^3}{\partial x^1} + dx^2 \cdot \frac{\vec{\partial} x^3}{\partial x^2} + d\vartheta^1 \cdot \frac{\vec{\partial} x^3}{\partial \vartheta^1} + d\vartheta^2 \cdot \frac{\vec{\partial} x^3}{\partial \vartheta^2},$$

$$d\vartheta^3 = dx^1 \cdot \frac{\vec{\partial} \vartheta^3}{\partial x^1} + dx^2 \cdot \frac{\vec{\partial} \vartheta^3}{\partial x^2} + d\vartheta^1 \cdot \frac{\vec{\partial} \vartheta^3}{\partial \vartheta^1} + d\vartheta^2 \cdot \frac{\vec{\partial} \vartheta^3}{\partial \vartheta^2},$$

and the matrices

$$\begin{bmatrix} \dfrac{\vec{\partial} x^3}{\partial x^1} & \dfrac{\vec{\partial} \vartheta^3}{\partial x^1} \\[2ex] \dfrac{\vec{\partial} x^3}{\partial \vartheta^1} & \dfrac{\vec{\partial} \vartheta^3}{\partial \vartheta^1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dfrac{\vec{\partial} x^3}{\partial x^2} & \dfrac{\vec{\partial} \vartheta^3}{\partial x^2} \\[2ex] \dfrac{\vec{\partial} x^3}{\partial \vartheta^2} & \dfrac{\vec{\partial} \vartheta^3}{\partial \vartheta^2} \end{bmatrix}$$

are invertible. Hence, if we set

$$\varphi_1=dx^1\cdot\frac{\vec{\partial}x^3}{\partial x^1}+d\vartheta^1\cdot\frac{\vec{\partial}x^3}{\partial\vartheta^1},\quad \varphi_2=dx^2\cdot\frac{\vec{\partial}x^3}{\partial x^2}+d\vartheta^2\cdot\frac{\vec{\partial}x^3}{\partial\vartheta^2},\quad \varphi_3=dx^3,$$

$$\psi_1=dx^1\cdot\frac{\vec{\partial}\vartheta^3}{\partial x^1}+d\vartheta^1\cdot\frac{\vec{\partial}\vartheta^3}{\partial\vartheta^1},\quad \psi_2=dx^2\cdot\frac{\vec{\partial}\vartheta^3}{\partial x^2}+d\vartheta^2\cdot\frac{\vec{\partial}\vartheta^3}{\partial\vartheta^2},\quad \psi_3=d\vartheta^3,$$

then $\varphi_i=\psi_i=0$ are normal systems.

Let $\Gamma_{\mathcal{W}}$ be the associated connection to the superweb $\mathcal{W}$. For convenience, we rewrite the functions x^i's and ϑ^i's as $z^1=x^1$, $z^2=x^2$, $z^3=\vartheta^1$, $z^4=\vartheta^2$, $w_1=x^3$ and $w_2=\vartheta^3$. Then the *Christoffel's symbols* $\{\Gamma^i_{jk}\}$ of $\Gamma_{\mathcal{W}}$ with respect to the local coordinate system z^1, z^2, z^3, z^4 are defined by

$$\vec{\nabla}_{\partial_j}(\partial_k)=\sum_{i=1}^{4}\Gamma^i_{jk}\cdot\partial_i,\qquad j,\ k=1,\ 2,\ 3,\ 4,$$

where ∂_j represents the vector field $\partial/\partial z^j$ and ∇ is the covariant differential operator of $\Gamma_{\mathcal{W}}$.

Let (g^{ij}) and (h^{ij}) be the inverse matrices of $\begin{pmatrix}\vec{\partial}_1w_1 & \vec{\partial}_1w_2\\ \vec{\partial}_3w_1 & \vec{\partial}_3w_2\end{pmatrix}$ and $\begin{pmatrix}\vec{\partial}_2w_1 & \vec{\partial}_2w_2\\ \vec{\partial}_4w_1 & \vec{\partial}_4w_2\end{pmatrix}$, respectively. Then Γ^i_{jk} are expressed as

$$\Gamma^{2i-1}_{2j-1\ 2k-1}=\sum_{r=1}^{2}(\vec{\partial}_{2j-1}\vec{\partial}_{2k-1}w_r)\cdot g^{ri}$$
$$-(-1)^{|2j-1|\cdot|2k-1|}\sum_{p,q,r=1}^{2}(\vec{\partial}_{2k-1}w_p)\cdot h^{pq}\cdot(\vec{\partial}_{2q}\vec{\partial}_{2j-1}w_r)\cdot g^{ri},$$

$$\Gamma^{2i}_{2j\ 2k}=\sum_{r=1}^{2}(\vec{\partial}_{2j}\vec{\partial}_{2k}w_r)\cdot h^{ri}$$
$$-(-1)^{|2j|\cdot|2k|}\sum_{p,q,r=1}^{2}(\vec{\partial}_{2k}w_p)\cdot g^{pq}\cdot(\vec{\partial}_{2q-1}\vec{\partial}_{2j}w_r)\cdot h^{ri},$$
$$i,\ j,\ k=1,\ 2,$$

where $|i|=0$ for $i=1$, 2 and $|i|=1$ for $i=3$, 4.

Remark 1. By using the above local expressions, we can easily express the components of the torsion and the curvature of $\Gamma_{\mathcal{W}}$. Then it can be seen that the torsion has four even and four odd components that are non trivial. Also the cuvature has sixteen even and sixteen odd non trivial components.

§ 7. Induced webs. To each even tensor field K on a supermanifold, we can associate a tensor field $\tilde{K}$ of the same type as K on the *body* of the supermanifold in a natural manner (cf. [3], [5]). Hence, a superweb $\mathcal{W}=\{\mathcal{D}_i\}$ on M induces a web $\tilde{\mathcal{W}}=\{\tilde{\mathcal{D}}_i\}$ on the body M_b of M. More precisely, if $\varphi_i=\psi_i=0$ is a system of local equations for $\mathcal{D}_i$, then $\tilde{\mathcal{D}}_i$ is defined by $\tilde{\varphi}_i=0$, $i=1$, 2, 3, respectively.

It is easily verified that the diffeomorphism induced from an equivalence of superwebs is also an equivalence of induced webs (cf. [5]). Thus we obtain the following theorem.

Theorem 3. *If two superwebs are equivalent, then the induced webs are equivalent.*

Moreover, the correspondence between superwebs and webs induces that of differential invariants of them: First, we note that the connection $\Gamma_{\mathcal{W}}$

associated to a superweb $\mathcal{W}$ induces a linear connection $\tilde{\Gamma}_{\mathcal{W}}$ on the body in a natural manner. Then we have the following theorem.

Theorem 4. *The induced connection $\tilde{\Gamma}_{\mathcal{W}}$ coincides with the connection associated to the induced web $\tilde{\mathcal{W}}$, and its curvature is induced from the curvature R of $\Gamma_{\mathcal{W}}$.*

Remark 2. Although the torsion T of $\Gamma_{\mathcal{W}}$ may not be zero, it can be shown that $\tilde{T}=0$. In fact, the even components of T and R correspond to the components of $\tilde{T}$ and $\tilde{R}$, respectively. Moreover, the four even components of T are sent to zero and only two of the sixteen even components of R remain alive under the correspondence and coincide with that of $\tilde{R}$.

References

[1] M. Batchelor: Two approaches to supermanifolds. Trans. Amer. Math. Soc., **258**, 257–270 (1980).

[2] W. Blaschke and G. Dubourdieu: Invarianten von Kurvengeweben. Hambg. Abh., **6**, 198–215 (1928).

[3] B. Kostant: Graded Manifolds, Graded Lie Theory And Prequantization. Lecture Note in Math. Springer., **270**, 177–306 (1977).

[4] I. M. Singer and S. Sternberg: On the infinit group of Lie and Cartan. I. Ann. Inst. Fourier., **15**, 1–114 (1965).

[5] M. Fujio: *G*-structures on supermanifolds. Master thesis, Osaka Univ. (1986) (in Japanese).