92. *Moduli Spaces of B_2-connections over Quaternionic Kähler Manifolds*

By Takashi NITTA

Department of Mathematics, Osaka University

(Communicated by Kunihiko KODAIRA, M. J. A., Oct. 12, 1987)

The purpose of this note is to announce our recent results on the moduli space of B_2-connections over a quaternionic Kähler manifold. Let M be a $4n$-dimensional compact, connected quaternionic Kähler manifold of positive scalar curvature, and $p: Z \to M$ the corresponding twistor space (see Salamon [8] for the definition of quaternionic Kähler manifolds and the corresponding twistor spaces). Furthermore, let E be a C^∞ complex vector bundle of rank r over M and h a Hermitian metric on E. In [6], we introduced the notion of B_2-connections on E, which generalizes that of anti-self-dual connections for $n=1$. We now define the sets $\mathcal{C}_B, \mathcal{C}_E$ and $\tilde{\mathcal{C}}_E$ as follows. $\mathcal{C}_B$: the set of all Hermitian, B_2-connections on (E, h) over M, $\mathcal{C}_E$: the set of all Einstein-Hermitian connections on (p^*E, p^*h) over Z, $\tilde{\mathcal{C}}_E$: the set of all Einstein-Hermitian connections D on (p^*E, p^*h) over Z, satisfying the following conditions (a) and (b).

(a) Write D as a sum of $D'+D''$ of its (1, 0)- and (0, 1)- components. (In terms of D'', the vector bundle $(F(=p^*E), p^*h)$ is a holomorphic vector bundle.) Then on each fibre Z_m $(m \in M)$ of $p: Z \to M$, the restricted vector bundle $(F|_{Z_m}, p^*h|_{Z_m})$ is a flat holomorphic vector bundlle. (Hence the real structure $\tau: Z \to Z$ (cf. Nitta and Takeuchi [7]) naturally lifts to a C^∞ bundle automorphism $\tau': F \to F$.)

(b) Let $\sigma: F \to F^*$ be the bundle map defined fibrewise by

$$F_z \ni f \longrightarrow \sigma(f) \in F^*_{\tau(z)} \qquad (z \in Z),$$

where $\sigma(f)(g) := (p^*h)(g, \tau'(f))$ for each $g \in F_{\tau(z)}$. Then σ is an anti-holomorphic bundle automorphism.

In [6; (0.2), (4.2)], we obtained the following generalization of a result of Atiyah, Hitchin and Singer [1].

Theorem 1. *The mapping*

$$p^*: \mathcal{C}_B \ni D \longrightarrow p^*D \in \tilde{\mathcal{C}}_E (\subset \mathcal{C}_E)$$

gives a bijective correspondence: $\mathcal{C}_B \simeq \tilde{\mathcal{C}}_E$.

The frame bundle of the Hermitian vector bundle (E, h) can be reduced to a principal $U(r)$-bundle P. Put $U_P := P \times_\theta U(r)$ and $\mathfrak{su}_P := P \times_{Ad'} \mathfrak{su}(r)$, where $\theta: U(r) \to \mathrm{Aut}\,(U(r))$ is the group conjugation and $Ad': U(r) \to GL(\mathfrak{su}(r))$ is the restriction of the adjoint representation of $U(r)$ to $\mathfrak{su}(r)$. The pull-back $p^*(U_P)$ and $p^*(\mathfrak{su}_P)$ are equal to $(p^*P) \times_\theta U(r)$ and $(p^*P) \times_{Ad'} \mathfrak{su}(r)$, respectively. Let D be a Hermitian B_2-connection on (E, h). Then there is an elliptic complex Σ_D on M, $\mathfrak{su}_P$-valued, introduced by Capria and Salamon [2; Theorem 3] and Nitta [6; (3.5)]. We denote the corresponding i-th cohomology group by $H^i(M, \Sigma_D)$. Now, a C^∞-section of U_P over M is called a *gauge transformation* of P. All gauge transformations of P form the *gauge group* $\mathcal{G}_P$ of P. Let C'_B be the set of all Hermitian B_2-connections D on E such that $H^0(M, \Sigma_D)=\{0\}$. The quotient space $\mathcal{B}'(:=C'_B/\mathcal{G}_P)$ is called the *moduli space* of irreducible Hermitian B_2-connections on (E, h). We then have:

Theorem 2. *The moduli space $\mathcal{B}'$ has a natural real analytic structure.*

Let C''_B be the set of all Hermitian B_2-connections D on (E, h) such that $H^0(M, \Sigma_D)=H^2(M, \Sigma_D)=\{0\}$, and let $\mathcal{B}''$ be the quotient space $C''_B/\mathcal{G}_P$. We next have:

Theorem 3. *The quotient space $\mathcal{B}''$ is a smooth manifold with a natural Riemannian metric.*

Let $\tilde{D}$ be an Einstein-Hermitian connection on (p^*E, p^*h). Then we have an elliptic complex $\Sigma_{\tilde{D}}$ on Z, $p^*(\mathfrak{su}_P)$-valued, introduced by Kim [4]. We denote the corresponding i-th cohomology group by $H^i(Z, \Sigma_{\tilde{D}})$. Now a C^∞-section of $p^*(U_P)$ over Z is called a *gauge transformation* of p^*P. All gauge transformations of p^*P form the *gauge group* $\mathcal{G}_{p^*P}$ of p^*P. Let C'_E be the set of all Einstein-Hermitian connections $\tilde{D}$ on (p^*E, p^*h) such that $H^0(Z, \Sigma_{\tilde{D}})=\{0\}$. Then the quotient space $\mathcal{E}':=C'_E/\mathcal{G}_{p^*P}$ is called the *moduli space* of irreducible Einstein-Hermitian connections on $(p^*E, p^*\mathrm{h})$. Recall the following theorem of Lübke and Okonek [5]:

Theorem 4. *The moduli space $\mathcal{E}'$ has a structure of a complex analytic space.*

Let C''_E be the set of all Einstein-Hermitian connections $\tilde{D}$ on (p^*E, p^*h) such that $H^0(Z, \Sigma_{\tilde{D}})=H^2(Z, \Sigma_{\tilde{D}})=\{0\}$ and let $\mathcal{E}''$ be the quotient space $C''/\mathcal{G}_{p^*P}$. We now quote the following result by Itoh [3] and Kim [4]:

Theorem 5. *The quotient space $\mathcal{E}''$ is a complex manifold with a natural Kähler metric.*

From Theorem 1, Theorem 2 and Theorem 4, we obtain:

Theorem 6. *$\mathcal{E}'$ admits a natural real structure. Furthermore, the moduli space $\mathcal{B}'$ is embedded in the fixed point set in $\mathcal{E}'$ of the real structure.*

Finally, combining Theorem 1 with Theorem 3 and Theorem 5, we obtain:

Theorem 7. *The quotient space $\mathcal{B}''$ is embedded in $\mathcal{E}''$ as a totally real submanifold and $\dim_R \mathcal{B}''=\dim_C \mathcal{E}''$.*

References

[1] M. F. Atiyah, N. J. Hitchin, and I .M. Singer: Self-duality in four-dimensional Riemannian geometry. Proc. Roy. Soc. London, Ser. A, **362**, 425–461 (1978).

[2] M. M. Capria and S. M. Salamon: Yang-Mills fields on quaternionic spaces (preprint).

[3] M. Itoh: The moduli space of Yang-Mills connections over a Kähler surface is a complex manifold. Osaka J. Math., **22**, 845–862 (1985).

[4] H. Kim: Moduli of Hermite-Einstein vector bundles (to appear in Math. Zeitschrift).
[5] M. Lübke and C. Okonek: Moduli of simple semi-connections and Einstein-Hermitian bundles (preprint).
[6] T. Nitta: Vector bundles over quaternionic Kähler manifolds (to appear in Tohoku Math. J.).
[7] T. Nitta and M. Takeuchi: Contact structures on twistor spaces. J. Math. Soc. Japan, **39**, 139–162 (1987).
[8] S. M. Salamon: Quaternionic Kähler manifolds. Inv. Math., **67**, 143–171 (1982).