90. *Graphs with Given Countable Infinite Group*

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§1. **Introduction.** In this note we shall prove the following

Theorem. *Let Δ be any graph which is a constant link of a finite graph and which has at least one isolated vertex and at least three vertices. Then for any countable group G there are infinitely many connected graphs Γ with constant link Δ and* $\mathrm{Aut}\,\Gamma\cong G$.

Since nK_1 is the constant link of $K_{n,n}$, we have the following

Corollary. *For any countable group G and any integer $n\geqq 3$ there are infinitely many connected n-regular graphs Γ with* $\mathrm{Aut}\,\Gamma\cong G$.

The case $n=3$ and with finite group G of this corollary was proved by Frucht [1] and this result was extended to general $n\geqq 3$ by Sabidussi [2]. The case with finite group G of our theorem was proved by Vogler [4]. Our proof is an extension of [4]. We shall use the same notations as in [4].

§2. *Proof of Theorem.* First we refer to the following lemma without proof, whose proof is similar to that in [4; Theorem 1].

Lemma 1. *Let G be a countable group, Δ a constant link of a finite graph with at least three vertices and at least one isolated vertex. If for each $k=3,4,5$ there are infinitely many connected k-regular prime graphs $\prod_k$ with* $\mathrm{Aut}\,\prod_k\cong G$ *and a stable k-coloring, then there are infinitely many connected graphs Γ with constant link Δ and* $\mathrm{Aut}\,\Gamma\cong G$.

Thus it is sufficient to prove the next lemma to prove our theorem.

Lemma 2. *Let G be a countable group. Then for each $k=3,4,5$ there are infinitely many connected k-regular prime graphs $\prod_k$ with* $\mathrm{Aut}\,\prod_k\cong G$ *and a stable k-coloring.*

Proof. First we show that for each $k=3, 4, 5$ there is a connected k-regular prime graph Γ_k with $\mathrm{Aut}\,\Gamma_k\cong G$ and a stable k-coloring. If G is generated by a finite number of its elements, we see the existence of such a graph Γ_k for each $k=3,4,5$ by graphs similarly constructed to those in [1; Theorem 4.1], [2; Theorem 3.7] and [4; Lemma 5]. So we assume for a while that G is not generated by any finite subset. Let $S=\{x_i : i\in N\}$ be an infinite subset of G satisfying $S\not\ni 1$ and $\langle S\rangle=G$. Let us set $G_i=\langle x_1, x_2, \cdots, x_i\rangle$. Now for every integer $i\geqq 2$ if x_i is contained in G_{i-1}, we remove x_i from S. Consequently, $G=\langle S\rangle$ holds and there is no finite subset $\{s_i : i=1,2,\cdots,t\}$ of S satisfying $s_1^{\varepsilon_1}s_2^{\varepsilon_2}\cdots s_t^{\varepsilon_t}=1$ with $\varepsilon_j=\pm 1$. Hereafter we set $S=\{y_i : i\in N\}$. Let us define graphs Γ_3, Γ_4 and Γ_5 as follows:

$V(\Gamma_3)=V(\Gamma_4)=V(\Gamma_5)=\{(j,g) : j\in N,\ g\in G\}$,

$E(\Gamma_3)=\{[(1,g),(2,g)],\ [(1,g),(3,g)],\ [(1,g),(4,g)],\ [(2,g),(5,g)],$

$[(2, g), (6, g)]$, $[(3, g), (6, g)]$, $[(3, g), (7, g)]$, $[(4, g), (7, g)]$, $[(4, g), (10, g)]$, $[(5, g), (6, g)]$, $[(5, g), (8, g)]$, $[(7, g), (8, g)]$, $[(8, g), (9, g)] : g \in G\} \cup \{[(m+8, g), (m+10, g)]$, $[(2m+7, g), (2m+8, y_m g)] : m \in N,\ g \in G\}$,

$E(\Gamma_4) = \{[(1, g), (2, g)]$, $[(1, g), (3, g)]$, $[(1, g), (4, g)]$, $[(1, g), (6, g)]$, $[(2, g), (3, g)]$, $[(2, g), (7, g)]$, $[(2, g), (10, g)]$, $[(3, g), (4, g)]$, $[(3, g), (5, g)]$, $[(4, g), (5, g)]$, $[(4, g), (8, g)]$, $[(5, g), (6, g)]$, $[(5, g), (11, g)]$, $[(6, g), (9, g)]$, $[(6, g), (12, g)] : g \in G\} \cup$ $\{[(m+6, g), (m+12, g)]$, $[(3m+4, g), (3m+5, y_{3m-2}g)]$, $[(3m+4, g), (3m+6, y_{3m-1}g)]$, $[(3m+5, g), (3m+6, y_{3m}g)] :$ $m \in N,\ g \in G\}$,

$E(\Gamma_5) = \{[(1, g), (2, g)]$, $[(1, g), (3, g)]$, $[(1, g), (4, g)]$, $[(1, g), (5, g)]$, $[(1, g), (7, g)]$, $[(2, g), (3, g)]$, $[(2, g), (8, g)]$, $[(2, g), (10, g)]$, $[(3, g), (4, g)]$, $[(3, g), (6, g)]$, $[(3, g), (9, g)]$, $[(4, g), (5, g)]$, $[(4, g), (9, g)]$, $[(5, g), (6, g)]$, $[(5, g), (9, g)]$, $[(6, g), (7, g)]$, $[(6, g), (10, g)]$, $[(7, g), (10, g)]$, $[(8, g), (9, g)]$, $[(2, g), (11, g)]$, $[(4, g), (12, g)]$, $[(8, g), (13, g)]$, $[(9, g), (14, g)]$, $[(5, g), (15, g)]$, $[(7, g), (16, g)]$, $[(8, g), (17, g)]$, $[(10, g), (18, g)]$, $[(6, g), (19, g)]$, $[(7, g), (20, g)]$, $[(8, g), (21, g)]$, $[(10, g), (22, g)] : g \in G\} \cup$ $\{[(m+10, g), (m+22, g)]$, $[(4m+7, g), (4m+8, y_{6m-5}g)]$, $[(4m+7, g), (4m+9, y_{6m-4}g)]$, $[(4m+7, g), (4m+10, y_{6m-3}g)]$, $[(4m+8, g), (4m+9, y_{6m-2}g)]$, $[(4m+8, g), (4m+10, y_{6m-1}g)]$, $[(4m+9, g), (4m+10, y_{6m}g)] : m \in N,\ g \in G\}$.

Then Γ_k is a connected k-regular prime graph with a stable k-coloring ($k=3, 4, 5$). For each $h \in G$ if we define a bijection $\sigma_h : V(\Gamma_k) \to V(\Gamma_k)$ by $\sigma_h(j, g) = (j, gh)$, we can find $\mathrm{Aut}\,\Gamma_k = \{\sigma_h : h \in G\} \cong G$ for each $k=3$, 4, 5 (see [1; Theorem 4.1]). Thus we complete the first paragraph of the proof.

Next we show that there are infinitely many (non-isomorphic) such graphs. We construct Γ_3' from Γ_3 by replacing each vertex by a triangle. Then we see that Γ_3' is a connected 3-regular prime graph with $\mathrm{Aut}\,\Gamma_3' \cong G$ and a stable 3-coloring (see [4; Lemma 4]). Of course Γ_3 and Γ_3' are not isomorphic to each other, because respective minimum circuits of Γ_3 and Γ_3' whose lengths are not divisible by three have different lengths. Hence we get infinitely many graphs $\prod_3$ as desired by repeating the above construction. Similarly if we construct Γ_4' from Γ_4 by replacing each vertex by a wheel with five vertices (i.e., $K_1 + C_4$) and Γ_5' from Γ_5 by replacing each vertex by a K_5 and if we repeat these constructions, then we get infinitely many graphs $\prod_4$ and infinitely many graphs $\prod_5$ as desired.

References

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