88. Boundly Factorizable $\mathcal{S}$-indecomposable Semigroups Generated by Two Elements

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1. Introduction. A semigroup S is called *$\mathcal{S}$-indecomposable* if it has no semilattice homomorphic image except the trivial image. In particular S is called *boundly factorizable* if $\bigcap_{n=1}^{\infty} S^n=\emptyset$, equivalently, for every $x\in S$ there is a positive integer n such that $x=x_1\cdots x_m$, $x_i\in S$ $(i=1,\cdots,m)$ implies $m\leq n$. Examples of such a semigroup are idempotent-free commutative archimedean semigroups [4], the semilattice components of a free semigroup [3], and so on. To study a boundly factorizable $\mathcal{S}$-indecomposable semigroup S generated by two elements, we consider S as a homomorphic image of the free semigroup of rank 2. Thus our study is reduced to the study of generating relations. In this paper we will investigate basic relations which produce S. See the commutative case in [2].

2. Preliminaries. A semigroup S is called *finitely factorizable* if every element of S is factorized into the product of elements of S in a finitely many ways. By an *irreducible basis* of S we mean a non-empty subset B of S such that (i) S is generated by B, (ii) if $x\in B$ then $x\neq yz$ for all $y, z\in S$. It is known [5] that if $S\backslash S^2$ generates S then $S\backslash S^2$ is an irreducible basis.

Lemma 1 (Theorem 2.2 in [1]). *If a semigroup S is boundly factorizable, then S has an irreducible basis R.*

From the definition we easily have

Lemma 2. *If S is boundly factorizable, then $x\neq yx$, $x\neq xz$ and $x\neq yxz$ for all $x, y, z\in S$. Hence S has no idempotent and S has no minimal ideal.*

Assume a semigroup S is generated by two elements p, q i.e. $R=\{p,q\}$. Let $C(p)$ and $C(q)$ denote the cyclic subsemigroups of S generated by p and q respectively, and $C(p,q)$ the subsemigroup of S consisting of all elements of S which can be expressed as the product of both p and q. In [3] $C(p)$, $C(q)$ and $C(p,q)$ were called contents.

Lemma 3. [3] *Every content in a semigroup is $\mathcal{S}$-indecomposable.*

Lemma 4. *S is $\mathcal{S}$-indecomposable if and only if $C(p)\cap C(p,q)\neq\emptyset$ and $C(q)\cap C(p,q)\neq\emptyset$.*

Let $o_p(v)$ denote the highest degree of the powers of p in v, for example, if $v=p^3qp^2q^2$, then $o_p(v)=3$ and $o_q(v)=2$.

Lemma 5. *If S is a boundly factorizable $\mathcal{S}$-indecomposable semigroup generated by p and q $(p\neq q)$, then*

(5.1) *there are positive integers* m, n *and elements* v, w *in* $C(p,q)$ *such that* $p^m=v$ *and* $q^n=w$ *where* $m>\max\{o_p(v), o_p(w)\}$ *and* $n>\max\{o_q(v), o_q(w)\}$.

(5.2) S *is finitely factorizable.*

Proof. (5.1) By Lemma 4, there are positive integers m, n and v, $w\in C(p,q)$ such that $p^m=v$ and $q^n=w$. Clearly $m>o_p(v)$. If $m\leq o_p(w)$ and $n\leq o_q(v)$, replace q^n in v by w; then it contradicts Lemma 2, therefore $m>o_p(w)$ or $n>o_q(v)$. Consider the case $m\leq o_p(w)$ and $n>o_q(v)$. Replacing p^k $(k\geq m)$ in w by $p^{k-m}v$ successively, it follows that $q^n=w'$, $m>o_p(w')$ for some $w'\in C(p,q)$. In the case where $o_q(v)\geq n$ and $o_p(w)<m$, we have similarly that $p^m=v'$, $n>o_q(v')$, $v'\in C(p,q)$. (5.2) Let $x\in S$ and assume that $x=x_1\cdots x_m$, $x_i\in S$ $(i=1,\cdots,m)$. Since S is generated by p and q, $x=y_1\cdots y_k$ where y_i is either p or q $(i=1,\cdots,k)$. By the assumption there is a positive integer n determined by x such that $m\leq k\leq n$. We can assume $n>1$. By elementary computation we see that the number of factorizations of x into the product of elements of S does not exceed

$$\sum_{i=2}^{n} 2^i(2^{i-1}-1).$$

3. Basic relations. A relation $p^m=v$ is called *satisfactory* if $m>o_p(v)$. A relation system $\{p^m=v,\ q^n=w\}$ is called *satisfactory* if both $p^m=v$ and $q^n=w$ are satisfactory and if $m>o_p(w)$ or $n>o_q(v)$. If "or" is replaced by "and", we say $\{p^m=v,\ q^n=w\}$ is *very satisfactory.* A satisfactory relation $p^m=v$ is called a *basic relation* if the following holds: If $p^{m'}=v'$ is a satisfactory relation and if $p^{m'}=v'$ implies $p^m=v$, then $m'=m$ and $v'=v$. A satisfactory relation system $\{p^m=v,\ q^n=w\}$ is called a *basic relation system* if the following holds: If $\{p^{m'}=v',\ q^{n'}=w'\}$ is a satisfactory relation system and if $\{p^{m'}=v',\ q^{n'}=w'\}$ implies $\{p^m=v,\ q^n=w\}$, then $m'=m$, $v'=v$, $n'=n$ and $w'=w$.

Lemma 6. *Assume* $\{p^m=v,\ q^n=w\}$ *is very satisfactory. This is a basic relation system if and only if both* $p^m=v$ *and* $q^n=w$ *are basic relations.*

Proof. The "only if" part. Assume that $p^{m'}=v'$ is satisfactory and $p^{m'}=v'$ implies $p^m=v$. Then $\{p^{m'}=v',\ q^n=w\}$ is satisfactory and $\{p^{m'}=v',\ q^n=w\}$ implies $\{p^m=v,\ q^n=w\}$. Hence $m'=m$, $v'=v$, so $p^m=v$ is a basic relation. Similarly $q^n=w$ is basic. The "if" part. This is immediate from the definitions.

By Lemmas 5 and 6 our study is reduced to that of basic relations.

Theorem 7. *Assume* $p^m=p^kvp^l$ *is a satisfactory relation where* v *has the form* qv_1q. *This is a basic relation if and only if one of the following* (1), (2), (3) *holds.*

(1) $k=l=0$.

(2) *If* $k>0$ *and* $l>0$, *then* $m=o_p(v)+1$ *or* $m=k+1=l+1$.

(3) *If* $k=0$ *and* $l>0$ *or if* $l=0$ *and* $k>0$, *then* $m=o_p(v)+1$.

Proof. Case (1). Nothing to say. Case (2) $k>0$ and $l>0$. Suppose $p^m=p^kvp^l$ is a basic relation. Then, for any i $(>k)$, $p^{m-i}=p^{k-i}vp^l$ is not satisfactory, hence $m-i\leq o_p(v)$ or $m-i\leq l$ for all $i(>k)$, equivalently,

$m \leq o_p(v)+1 < m+1$ or $m \leq l+1 < m+1$, so $m=o_p(v)+1$ or $m=l+1$. Since $p^{m-j}=p^k v p^{l-j}$ $(j<l)$ is not satisfactory, we have similarly that $m=o_p(v)+1$ or $m=k+1$. Thus we have the conclusion. Conversely, assume $m=o_p(v)+1$ and let $v'=qv_1'q$. If $p^{m'}=p^{k'}v'p^{l'}$ is satisfactory and if $p^{m'}=p^{k'}v'p^{l'}$ implies $p^m=p^k v p^l$, then $m-m'=k-k'+l-l'$, $k-k'\geq 0$, $l-l'\geq 0$, $v'=v$. On the other hand $m'>o_p(p^{k'}vp^{l'})=\max\{k', o_p(v), l'\}=m-1$, whence $0\leq m-m'<1$, so $m=m'$, $k=k'$, $l=l'$. Hence $p^m=p^k v p^l$ is basic. Assume that $m=k+1=l+1$ and that $p^{m'}=p^{k'}v'p^{l'}$ is satisfactory and if $p^{m'}=p^{k'}v'p^{l'}$ implies $p^m=p^k v p^l$. For the same reason as above it holds that $m'>k'$ and $m'>l'$. This implies that $l'=(k-k')+m'-1<m'$ and $k'=(l-l')+m'-1<m'$, and hence $0\leq k-k'<1$ and $0\leq l-l'<1$; then we have $k=k'$, $l=l'$, $m=m'$, thus $p^m=p^k v p^l$ is basic. Case (3). $k=0$ and $l>0$. Suppose $p^m=vp^l$ is a basic relation. Since $p^m=vp^l$ is satisfactory, it follows that $m>o_p(vp^l)=\max\{o_p(v), l\}$ hence $m>o_p(v)$ and $m>l$. If $m>o_p(v)+1$, then $m-1>\max\{o_p(v), l-1\}=o_p(vp^{l-1})$. It follows that $p^{m-1}=vp^{l-1}$ is satisfactory. However this contradicts the assumption that $p^m=vp^l$ is basic. Therefore $m\leq o_p(v)+1$, so $m=o_p(v)+1$. Similarly we have the same result in case $l=0$ and $k>0$. Conversely, let $m=o_p(v)+1$. If $p^{m'}=v'p^{l'}$ is satisfactory and if $p^{m'}=v'p^{l'}$ implies $p^m=vp^l$, then $m-m'=l-l'\geq 0$, $v'=v$ and $m'>\max\{o_p(v'), l'\}=m-1$. Hence $m=m'$, $l=l'$ and $p^m=vp^l$ is basic. The case $l=0$ and $k>0$ is similarly treated.

A boundly factorizable $\mathcal{S}$-indecomposable semigroup generated by two elements p, q is a homomorphic image of the free semigroup $F(p, q)$ relative to a basic relation system $\{p^m=u, q^n=w\}$ where $p^m=u$ and $q^n=w$ satisfy the condition given by Theorem 7.

References

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