82. *Uniqueness in the Characteristic Cauchy Problem under a Convexity Condition*

By Katsuju IGARI
Department of Applied Mathematics, Faculty of Engineering, Ehime University, Matsuyama, Ehime 790

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1987)

We consider the Cauchy problem with characteristic initial surface assuming the coefficients to be analytic. Though the uniqueness does not hold in general for C^∞ or $\mathcal{D}'$ solutions, we can expect it if we impose some convexity condition. We establish such a uniqueness theorem at a doubly characteristic point. The result makes us be able to understand the Trèves' example [6] in a general structure.

1. Result. Let U be a neighborhood of the origin in $\boldsymbol{R}^{n+1}$, $P(x;\partial)=\sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha$, $x=(x_0,\cdots,x_n)$, and $a_\alpha(x)$ be analytic functions in U. We denote the principal symbol of P by $p_m(x, \sum \xi_i dx_i)$. Let S be a hypersurface defined by $\varphi(x)=0$, where φ is a real-valued analytic function satisfying $\varphi(0)=0$ and $d\varphi\neq 0$ in U.

We assume

(A) $\qquad p_m(x, d\varphi)\equiv 0$ in U, and $dp_m(x, d\varphi)=0$ at $x=0$.

Under this assumption, we define

$$G=\left(\frac{\partial p_m^{(e_i)}(x, d\varphi)}{\partial x_j}(0)\,;\ \begin{matrix} i=0\downarrow n \\ j=0\rightarrow n\end{matrix}\right).$$

Let $\lambda_0,\cdots,\lambda_n$ be the eigen values of this matrix. Besides, we put

$$\mu=\left\{p_{m-1}(x, d\varphi)+\sum_{|\alpha|=2}\frac{1}{\alpha\,!}p_m^{(\alpha)}(x, d\varphi)\partial_x^\alpha\varphi\right\}_{x=0}.$$

Note. 1) These $n+2$ values $\lambda_0,\cdots,\lambda_n,\mu$ are invariant with respect to the change of coordinates.

2) The matrix G has at least one zero eigen value.

3) Let F be the fundamental matrix of p_m at its critical point $(0, d\varphi(0))$. Then, under the assumption (A), the eigen values of F are equal to $\{\pm\lambda_0,\cdots,\pm\lambda_n\}$, where λ_i's are those of G.

Now let k be the number of non-zero eigen values of G. We put the following four conditions:

C.1 $\qquad k\geq 1.$

C.2 Let Λ be the convex hull, on the complex number plane, of non-zero eigen values of G, then $0\notin\Lambda$.

C.3 $\qquad \mu\notin\left\{\sum_{i=0}^n \lambda_i\beta_i\,;\ \beta\in\boldsymbol{N}^{n+1}\right\}.$

C.4 There are n real-valued analytic functions $\varphi_i(x)$, $i=1,\cdots,n$, such that $d\varphi, d\varphi_1,\cdots,d\varphi_n$ are linearly independent and that

$$p_m=0 \quad \text{and} \quad dp_m=0$$

on $\{(x,\theta); \varphi_i(x)=0 \text{ and } \xi_i=0 \text{ for } i=1,\cdots,k\}\subset T^*U$, where $\theta=\xi_0 d\varphi_0+\cdots+\xi_n d\varphi_n$, $\varphi_0=\varphi$.

Then our result is:

Theorem. *Under the assumption* (A), *suppose four conditions* C.1, 2, 3 *and* 4. *Then* $u\in\mathcal{D}'(U)$, $Pu=0$ *and* $\operatorname{supp}[u]\subset\{x=0\}\cup\{x;\varphi(x)>0\}$ *imply* $0\notin\operatorname{supp}[u]$.

Note. In this note we call convexity condition the condition $\operatorname{supp}[u]\subset\{x=0\}\cup\{x;\varphi(x)>0\}$.

2. Example. Let us consider the operator

$$P_b=\partial_0^2-x_0^2\partial_1^2+b\partial_1,\ b \text{ constant}.$$

It has two phase functions $\varphi_\pm=(1/2)x_0^2\pm x_1$. The following 1)–3) hold:

1) There exists a solution $u\in C^\infty$ of $P_b u=0$ such that $(0,0)\in\operatorname{supp}[u]\subset\{x_1\geq-(1/2)x_0^2\}$.

2) If $b\notin\{1,3,5,\cdots\}$, then $u\in\mathcal{D}'$, $P_b u=0$ and $\operatorname{supp}[u]\subset\{(0,0)\}\cup\{x_1>-(1/2)x_0^2\}$ imply $(0,0)\notin\operatorname{supp}[u]$.

3) If $b\in\{1,3,5,\cdots\}$, then there exists a solution $u\in C^\infty$ of $P_b u=0$ such that $(0,0)\in\operatorname{supp}[u]\subset\{x_1\geq(1/2)x_0^2\}$.

This example essentially dues to F. Trèves [6], see also Birkland and Persson [1]. The uniqueness part 2) is a typical example of our theorem. More generally, let $0\leq k<m\leq n$ and

$$P_{ab}=\partial_0^2+\cdots+\partial_k^2-(a_0^2x_0^2+\cdots+a_k^2x_k^2)(\partial_{k+1}^2+\cdots+\partial_m^2)+b_0\partial_0+\cdots+b_n\partial_n,$$

where a_i are positive constants and b_i are analytic functions. Let $\psi(x_{k+1},\cdots,x_n)$ be a real-valued analytic function which satisfies $(\partial_{k+1}\psi)^2+\cdots+(\partial_m\psi)^2\equiv 1$ and $\psi(0)=0$. Then

$$\varphi=\frac{1}{2}a_0x_0^2+\cdots+\frac{1}{2}a_kx_k^2+\psi(x_{k+1},\cdots,x_n)$$

is a phase function of P_{ab}. If we suppose

$$\sum_{i=k+1}^{n}(b_i\partial_i\psi)(0)\notin\left\{\sum_{i=0}^{k}(2\beta_i+1)a_i;\ \beta_i\in N\cup\{0\}\right\},$$

then P_{ab} and φ satisfy all the required conditions and consequently our uniqueness theorem holds for them.

3. Remarks. 1) The proof of the theorem is done in a parallel way as that of the Holmgren's uniqueness theorem. We first consider the Cauchy problem for the transposed equation ${}^tPu=f$ with initial data on the hypersurface $\varphi=c$. We note that this hypersurface is characteristic to the operator ${}^tP$. Given $m-1$ initial data, we establish an existence and uniqueness theorem in the category of holomorphic functions, cf. [2]. It is important to see that the size of the existence domain of solution does not depend particularly on the small parameter c. We can then prove the theorem in a standard way. The details will be given in our forthcoming paper.

2) The uniqueness at a characteristic point is closely related to the propagation of analytic wave front sets. Uuiqueness like in the theorem

and sometimes sharper one follows from the invariancy of $WF_a(u)$ along the bicharacteristic strips, see Sjöstrand [5] and its references, where the operators of principal type and those having involutory characteristics are studied. T. Oaku [4] investigated a certain class of operators having non-involutory characteristics. Particularly, when $k=0$, our uniqueness theorem for the operator P_{ab} follows from his result.

References

[1] B. Birkland and J. Persson: The local Cauchy problem in $\boldsymbol{R}^2$ at a point where two characteristic curves have a common tangent. J. Diff. Eq., **30**, 64–88 (1978).

[2] K. Igari: On the Cauchy-Kowalewski theorem for characteristic initial surfaces. Proc. Japan Acad., **63**A, 7–9 (1987).

[3] L. J. Korbly: Uniqueness of solutions of partial differential equations when the initial surface is characteristic at a point. Proc. A.M.S., **86**, 617–624 (1982).

[4] T. Oaku: A canonical form of a system of microdifferential equations with non-involutory characteristics and branching of singularities. Invent. math., **65**, 491–525 (1982).

[5] J. Sjöstrand: Singularités analytiques microlocales. Astérisque, 95 (1982).

[6] F. Trèves: Discrete phenomena in uniqueness in the Cauchy problem. Proc. A.M.S., **46**, 229–233 (1974).