81. A Convergence of Solutions of an Inhomogeneous Parabolic Equation

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Our aim in this paper is to prove that a *compensated solution* (see (4)) of an inhomogeneous parabolic equation (1) converges to a classical solution of an elliptic equation (2) as $t\to\infty$.

(1) $\partial_t u=(A+\sum_{|\alpha|=2q} B_\alpha(x)\partial^\alpha)u+f(x),\ t>0,\ x\in \boldsymbol{R}^d;\ u(0,x)=0.$

(2) $(A+\sum_{|\alpha|=2q} B_\alpha(x)\partial^\alpha)v+f(x)=c_f,\qquad x\in \boldsymbol{R}^d.$

Where

$$A\equiv(-1)^{q-1}\rho\sum_{k=1}^d \frac{\partial^{2q}}{\partial x_k^{2q}}$$

with a natural number q and a complex number ρ such that $\mathrm{Re}\,\rho>0$; $B_\alpha(x)$'s are functions in a certain class $\mathcal{F}^0(\boldsymbol{R}^d)$ and *"smaller"* than $\mathrm{Re}\,\rho$; $f(x)$ is in a class $\mathcal{F}^0(\boldsymbol{R}^d)$; and c_f is a constant determined from f.

As easily seen, the solution u of (1) possibly blows up as $t\to\infty$ (see after Proposition 2). Hence we shall consider the *compensated solution* $\tilde{u}$ instead of u itself. $\tilde{u}$ is written by a *Girsanov type formula* given in [1], [2], and it enables us to prove that $\tilde{u}$ converges to solution of (2).

1. We shall state the notations briefly. More precise descriptions can be found in [1], [2].

For multiindex α and $x\in\boldsymbol{R}^d$, we put

$$x^\alpha\equiv\prod_{k=1}^d x_k^{\alpha_k}\quad\text{and}\quad \partial^\alpha\equiv\prod_{k=1}^d\left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}.$$

For a non-negative number κ, $\mathcal{M}^\kappa(\boldsymbol{R}^d)$ is a Banach space consisting of all complex valued measures $\mu(d\xi)$ on $\boldsymbol{R}^d$ with $\|\mu\|_\kappa\equiv\int(1+|\xi|)^\kappa|\mu|(d\xi)<\infty$, and $\mathcal{F}^\kappa(\boldsymbol{R}^d)$ is a Banach space of all Fourier transforms of $\mathcal{M}^\kappa(\boldsymbol{R}^d)$, i.e. $f(x)=\int\exp\{i\xi\cdot x\}\mu_f(d\xi)$, $\mu_f\in\mathcal{M}^\kappa(\boldsymbol{R}^d)$, and we define as $\|f\|_\kappa\equiv\|\mu_f\|_\kappa$. $f\in\mathcal{F}^0(\boldsymbol{R}^d)$ is bounded and uniformly continuous, and $\sup_x|f(x)|\leqq\|f\|_0$.

Put $\boldsymbol{R}^+\equiv(0,\infty)$, and $\mathcal{M}^\kappa(\boldsymbol{R}^+,\boldsymbol{R}^d)$ denotes a set of all complex valued measures $\mu(t,d\xi)$, $t\in\boldsymbol{R}^+$, such that (i) $\mu\in\mathcal{M}^\kappa(\boldsymbol{R}^d)$ for each $t\in\boldsymbol{R}^+$, and (ii) $\|\mu(t,\cdot)-\mu(s,\cdot)\|_\kappa\to0$ as $t\to s$ on $\boldsymbol{R}^+$. $\mathcal{F}^\kappa(\boldsymbol{R}^+,\boldsymbol{R}^d)$ is a space consisting of all Fourier transforms of $\mathcal{M}^\kappa(\boldsymbol{R}^+,\boldsymbol{R}^d)$, i.e.

$$g(t,x)=\int\exp\{i\xi\cdot x\}\mu_g(t,d\xi),\qquad \mu_g\in\mathcal{M}^\kappa(\boldsymbol{R}^+,\boldsymbol{R}^d).$$

2. By a slight modification of the argument in [2], we get:

Proposition 1. *Assume that* (i) f *and* B_α's *on* (1) *are in* $\mathcal{F}^0(\boldsymbol{R}^d)$, *and* (ii) $\Sigma_{|\alpha|=2q}\|B_\alpha\|_0<\mathrm{Re}\,\rho$. *Then* (1) *possesses a unique classical solution u such* *that* $\partial_t u$, $\partial^a u \in \mathcal{F}^0(\boldsymbol{R}^+, \boldsymbol{R}^d)$ *for* $|a| \leqq 2q$.

On the other hand, for a homogeneous parabolic equation

(3) $\quad \partial_t v = Av + \sum_{|a|=2q} B_a(x) \partial^a v, \quad t>0, \quad x \in \boldsymbol{R}^d; \quad v(0,x)=f(x),$

we know the following result (see [2, 3]):

Proposition 2. *Under the hypotheses in Proposition* 1, (3) *possesses a unique wide sense solution* v, *and* $\lim_{t\to\infty} \|v(t,\cdot)-c_f\|_0 = 0$ *for a constant* c_f.

The solution $u(t,x)$ of (1) is not necessarily finite, when t tends infinity. For instance, if $f(x)=c$ for a non zero constant c, then $u(t,x)=ct$, and $|u|\to\infty$ as $t\to\infty$.

Therefore, we introduce a *compensated solution* $\tilde{u}(t,x)$ instead of u itself:

(4) $$\tilde{u}(t,x) \equiv u(t,x) - \sum_{|a|\leqq 2q-1} \frac{x^a}{|a|!} \partial^a u(t,0).$$

Our assertion in this paper is the following.

Theorem. *Under the hypotheses in Proposition* 1, *as* $t\to\infty$, $\tilde{u}(t,x)$ *converges to a classical solution of* (2) *uniformly on compact sets, where* c_f *is a constant given in Proposition* 2.

Corollary. *If the measure* μ_f *corresponding to* f *is absolutely continuous in the Lebesgue measure, i.e.*

$$f(x) = \int \exp\{i\xi\cdot x\} \hat{f}(\zeta) d\zeta \qquad \text{for } \hat{f} \in L_1(\boldsymbol{R}^d),$$

then c_f *in Proposition* 2 *and Theorem is zero.*

3. We denote by $\mu_f(d\zeta)$ and $\nu_a(d\xi)$, $|a|=2q$, the measures corresponding to f and B_a's, respectively. Define

$$\langle y \rangle \equiv (\textstyle\sum_{k=1}^d y_k^{2q})^{1/2q} \qquad \text{for } y \in \boldsymbol{R}^d,$$

$$H(1) \equiv \zeta \quad \text{and} \quad H(j) \equiv \zeta + \xi^{(1)} + \cdots + \xi^{(j-1)} \qquad \text{for } j \geqq 2.$$

Let u be the solution of (1) given in Proposition 1, and let v be that of (3) in Proposition 2, then $u(t,x) = \int_0^t v(s,x)ds$. As in [2], [3], we can write

(5) $$\partial_t u(t,x) = v(t,x) = \int \mu_f(d\zeta) \exp\{i\zeta\cdot x - \rho\langle\zeta\rangle^{2q} t\} + \sum_{n=1}^\infty \sum_{|a^{(1)}|=2q} \cdots \sum_{|a^{(n)}|=2q} I(t,x; a^{(1)}, \cdots, a^{(n)}),$$

where, with the convention $s_0 \equiv t$,

(6) $$I(t,x; a^{(1)}, \cdots, a^{(n)}) \equiv \int_{t>s_1>\cdots>s_n>0} ds_1 \cdots ds_n \int \mu_f(d\zeta) \times \int \nu_{a^{(1)}}(s_1, d\xi^{(1)}) \cdots \int \nu_{a^{(n)}}(s_n, d\xi^{(n)}) \exp\{iH(n+1)\cdot x\} \times (\textstyle\prod_{j=1}^n (iH(j))^{a^{(j)}} \exp\{-\rho\langle H(j)\rangle^{2q}(s_{j-1}-s_j)\}) \exp\{-\rho\langle H(n+1)\rangle^{2q} s_n\}.$$

4. Using (5) and (6), we shall prove the theorem and the corollary in the following four steps.

Step 1. First we take a sequence $\{f^{(m)}\}$, $m=1,2,\cdots$, in $\mathcal{F}^{2q}(\boldsymbol{R}^d)$ such that $\|f-f^{(m)}\|_0 \to 0$ as $m\to\infty$. By Proposition 1, we have a classical solution $u^{(m)}$ of

(7) $\quad \partial_t u^{(m)} = Au^{(m)} + \sum_{|a|=2q} B_a \partial^a u^{(m)} + f^{(m)}; \qquad u^{(m)}(0,x)=0.$

$\{u^{(m)}\}$ converges to u, and $\partial_t \partial^a u^{(m)}$ are in $\mathcal{F}^0(\boldsymbol{R}^+, \boldsymbol{R}^d)$ for $|a| \leqq 2q$, since $f^{(m)}$

$\in \mathcal{F}^{2q}(\boldsymbol{R}^d)$. We define $\tilde{u}^{(m)}$ as (4) with $u^{(m)}$ in the place of u.

Step 2. We denote by $\mu_f^{(m)} \in \mathcal{M}^{2q}(\boldsymbol{R}^d)$ the corresponding measure to $f^{(m)}$, and define $I^{(m)}(t, x; a^{(1)}, \cdots, a^{(n)})$ as (6) with $\mu_f^{(m)}$ in the place of μ_f. Put

$$\tilde{I}^{(m)}(t, x; a^{(1)}, \cdots, a^{(n)}) \equiv I^{(m)}(t, x; a^{(1)}, \cdots, a^{(n)}) - \sum_{|\beta| \leqq 2q-1} \frac{x^\beta}{|\beta|!} \partial^\beta I^{(m)}(t, 0; a^{(1)}, \cdots, a^{(n)}),$$

and this makes sense, because $\mu_f^{(m)} \in \mathcal{M}^{2q}(\boldsymbol{R}^d)$. Noticing that $|y^a| \leqq \langle y \rangle^{2q}$ for $|a| = 2q$, we get

$$\int_0^\infty ds \sup_{|x| \leqq K} |\partial^\beta \tilde{I}^{(m)}(s, x; a^{(1)}, \cdots, a^{(n)})| \leqq C(1+K)^{2q} \frac{\|f^{(m)}\|_0}{(\mathrm{Re}\,\rho)^{n+1}} \|B_{a^{(1)}}\|_0 \cdots \|B_{a^{(n)}}\|_0, \qquad |\beta| \leqq 2q,$$

where C is a positive constant depending only on q and d. Put $\theta \equiv \sum_{|a|=2q} \|B_a\|_0 / \mathrm{Re}\,\rho$, then (4) through (6) derive

$$(8) \qquad \int_0^\infty ds \sup_{|x| \leqq K} |\partial_t \partial^\beta \tilde{u}^{(m)}(s, x)| \leqq \frac{C(1+K)^{2q} \|f^{(m)}\|_0}{\mathrm{Re}\,\rho(1-\theta)}, \qquad |\beta| \leqq 2q.$$

Now $\tilde{u}^{(m)}(t, x)$, together with the special derivatives up to the order $2q$, converges to a certain function $\tilde{u}_\infty^{(m)}(x)$ uniformly on compact sets as $t \to \infty$, because

$$\sup_{|x| \leqq K} |\partial^\beta \tilde{u}^{(m)}(T, x) - \partial^\beta \tilde{u}^{(m)}(T', x)| = \int_{T'}^T ds \sup_{|x| \leqq K} |\partial_t \partial^\beta \tilde{u}^{(m)}(s, x)|, \qquad |\beta| \leqq 2q,$$

on which (8) implies that the right hand side vanishes as $T, T' \to \infty$.

Step 3. We make a similar calculation as in Step 2, and get

$$(9) \qquad \sup_{|x| \leqq K} |\partial^\beta \tilde{u}^{(m)}(t, x) - \partial^\beta \tilde{u}(t, x)| \leqq \frac{C(1+K)^{2q} \|f^{(m)} - f\|_0}{\mathrm{Re}\,\rho(1-\theta)}$$

for $|\beta| \leqq 2q$. In addition, we also have

$$(10) \qquad \sup_{t>0} \|\partial_t u^{(m)}(t, \cdot) - \partial_t u(t, \cdot)\|_0 \leqq \frac{C \|f^{(m)} - f\|_0}{\mathrm{Re}\,\rho(1-\theta)}.$$

Since $\partial_t u = v$, (10) and Proposition 2 yield

$$(11) \qquad \lim_{t, m \to \infty} \|\partial_t u^{(m)}(t, \cdot) - c_f\|_0 = 0 \qquad \text{for a constant } c_f.$$

Noticing that $\partial^\beta \tilde{u}^{(m)} = \partial^\beta u^{(m)}$ for $|\beta| = 2q$, we let $t, m \to \infty$ on (7). Then the theorem follows from a combination of the conclusion at Step 2 with (9) and (11).

Step 4. As in [3], the hypothesis on the corollary implies that $c_f = 0$ on Proposition 2, and the proof is completed.

References

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