79. *Characterization of the Eigenfunctions in the Singularly Perturbed Domain*

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The eigenvalue problem of the Laplace operator and its relation to the geometrical structure of the domain have been studied by many authors. It is usually the case that the eigenvalue and its eigenfunction vary continuously under the smooth deformation of the domain. But if the deformation of the domain is weak or wild (i.e. the topological type is not conserved or some part of the domain degenerates, etc.), the set of the eigenvalues may not converge to that of the limit domain and some of the eigenfunctions may behave singularly on the perturbed portion of the domain. We deal with a singularly perturbed domain $\Omega(\zeta)=D_1\cup D_2\cup Q(\zeta)$ ($\zeta>0$) where one part $Q(\zeta)$ degenerates to a one-dimensional segment as the parameter $\zeta\to 0$ while $\bar{D}_1\cap\bar{D}_2=\emptyset$ and we characterize the behavior of the eigenfunctions. J. T. Beale [1] has dealt with a domain perturbation similar to the above. He has considered an exterior domain $\mathcal{D}$ of a bounded obstacle with a partially open cavity and characterized the set of the scattering frequencies when the channel to the cavity is sufficiently narrow. Especially in the case of Neumann boundary condition, he has proved that the set of the scattering frequencies is approximated by the union of the scattering frequencies and that of the eigenfrequencies on the limit line segment of the channel $\mathcal{N}$ with the Dirichlet boundary condition on the endpoints of the segment. See [1] for details. By applying the method of [1] to our situation, the set of the eigenvalues $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ of $-\Delta$ for the Neumann boundary condition, is decomposed as follows: $\{\mu_k(\zeta)\}_{k=1}^{\infty}=\{\lambda_k(\zeta)\}_{k=1}^{\infty}\cup\{\omega_k(\zeta)\}_{k=1}^{\infty}$ where $\{\omega_k(\zeta)\}_{k=1}^{\infty}$ approximates the set of the eigenvalues on $D_1\cup D_2$ and $\{\lambda_k(\zeta)\}_{k=1}^{\infty}$ approximates the set of the eigenvalues of $-(d^2/dz^2)$ in the limit line segment for the Dirichlet boundary condition. The purpose of this paper is to present a characterization theorem for the eigenfunctions to $\{\lambda_k(\zeta)\}_{k=1}^{\infty}$ and $\{\omega_k(\zeta)\}_{k=1}^{\infty}$, respectively.

§1. **Formulation.** We specify the singularly perturbed domain $\Omega(\zeta)$ in $\boldsymbol{R}^n$ in the following form,

$$\Omega(\zeta)=D_1\cup D_2\cup Q(\zeta)$$

where $D_i(i=1,2)$ and $Q(\zeta)$ are defined in the following conditions where $x'=(x_2,x_3,\cdots,x_n)\in\boldsymbol{R}^{n-1}$.

(A.1) D_1 and D_2 are bounded domains in $\boldsymbol{R}^n$ (mutually disjoint) with smooth boundary which satisfy the following conditions for some positive constant $\zeta_*>0$.

$$\bar{D}_1 \cap \{x=(x_1, x') \in \boldsymbol{R}^n \mid x_1 \leqq 1, |x'|<3\zeta_*\}$$
$$=\{(1, x') \in \boldsymbol{R}^n \mid |x'|<3\zeta_*\}$$
$$\bar{D}_2 \cap \{x=(x_1, x') \in \boldsymbol{R}^n \mid x_1 \geqq -1, |x'|<3\zeta_*\}$$
$$=\{(-1, x') \in \boldsymbol{R}^n \mid |x'|<3\zeta_*\}$$

(A.2) $Q(\zeta)=R_1(\zeta) \cup R_2(\zeta) \cup \Gamma(\zeta)$

$$R_1(\zeta)=\{(x_1, x') \in \boldsymbol{R}^n \mid 1-2\zeta<x_1 \leqq 1, |x'|<\zeta\rho((x_1-1)/\zeta)\}$$
$$R_2(\zeta)=\{(x_1, x') \in \boldsymbol{R}^n \mid -1 \leqq x_1<-1+2\zeta, |x'|<\zeta\rho((-1-x_1)/\zeta)\}$$
$$\Gamma(\zeta)=\{(x_1, x') \in \boldsymbol{R}^n \mid -1+2\zeta \leqq x_1 \leqq 1-2\zeta, |x'|<\zeta\}$$

where $\rho \in C^0((-2, 0]) \cap C^\infty((-2, 0))$ is a positive valued monotone increasing function such that $\rho(0)=2$, $\rho(s)=1$ for $s \in (-2, -1)$ and the inverse function ρ^{-1}: $[1,2] \rightarrow [-1, 0]$ satisfies $\lim_{\xi \to 2}(d^k\rho^{-1}/d\xi^k)(\xi)=0$ for any nonnegative integer k. Hereafter we denote the points $p_1=(1, 0, \cdots, 0)$, $p_2=(-1, 0, \cdots, 0)$ and the set $L=\bigcap_{0<\zeta<\zeta_*} \overline{Q(\zeta)}=\{(z, 0, \cdots, 0) \in \boldsymbol{R}^n \mid -1 \leqq z \leqq 1\}$.

We consider the following eigenvalue problem (1.1) in $\Omega(\zeta)$ for small $\zeta>0$.

$$\begin{cases} \Delta\Phi+\mu\Phi=0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial\Phi}{\partial\nu}=0 & \text{on } \partial\Omega(\zeta), \end{cases} \tag{1.1}$$

where $\Delta=\sum_{k=1}^n \partial^2/\partial x_k^2$ is the Laplace operator and ν denotes the unit outward normal vector on $\partial\Omega(\zeta)$. Let $\{\mu_k(\zeta)\}_{k=1}^\infty$ be the eigenvalues of (1.1) arranged in increasing order (counting multiplicity). By applying the method of J. T. Beale [1], we can separate the set of the eigenvalues of (1.1) for small $\zeta>0$, i.e. $\{\mu_k(\zeta)\}_{k=1}^\infty$ is expressed as follows

$$\{\mu_K(\zeta)\}_{k=1}^\infty=\{\omega_k(\zeta)\}_{k=1}^\infty \cup \{\lambda_k(\zeta)\}_{k=1}^\infty, \tag{1.2}$$

where $\{\omega_k(\zeta)\}_{k=1}^\infty$ is a perturbation of the set of the eigenvalues in $D_1 \cup D_2$ for the Neumann boundary condition and $\{\lambda_k(\zeta)\}_{k=1}^\infty$ is a perturbation of that for the operator $-(d^2/dz^2)$ in L with the Dirichlet boundary condition on the endpoints of the segment L.

Let $\{\omega_k\}_{k=1}^\infty$ and $\{\phi_k\}_{k=1}^\infty$ be respectively the sequence of the eigenvalues arranged in increasing order and the complete system of the corresponding orthonormalized eigenfunctions of the following eigenvalue problem in $D_1 \cup D_2$.

$$\begin{cases} \Delta\phi+\omega\phi=0 & \text{in } D_1 \cup D_2, \\ \dfrac{\partial\phi}{\partial\nu}=0 & \text{on } \partial D_1 \cup \partial D_2. \end{cases} \tag{1.3}$$

$$(0=\omega_1=\omega_2 \leqq \omega_3 \leqq \cdots \rightarrow \infty, \ \|\phi_k\|_{L^2(D_1 \cup D_2)}=1, \ k \geqq 1)$$

We put $\lambda_k=(k\pi/2)^2$ and $S_k(z)=\sin(k\pi/2)(z+1)$ $(k \geqq 1)$ which are respectively the eigenvalues and the complete system of the eigenfunctions of the following eigenvalue problem

$$\begin{cases} \dfrac{d^2}{dz^2}S+\lambda S=0, & -1<z<1, \\ S(-1)=S(1)=0. \end{cases} \tag{1.4}$$

We also assume the following condition

$$\{\lambda_k\}_{k=1}^\infty \cap \{\omega_k\}_{k=1}^\infty=\varnothing. \tag{A.3}$$

By the above argument, we have,

Proposition.

(1.5) $$\lim_{\zeta\to 0}\omega_k(\zeta)=\omega_k,\quad \lim_{\zeta\to 0}\lambda_k(\zeta)=\lambda_k\quad (k=1,2,3,\cdots).$$

§ 2. Main result.

Theorem. *Assume $n\geqq 3$. We can choose a complete system of the orthonormalized eigenfunctions $\{\Phi_{k,\zeta}\}_{k=1}^{\infty}$ ($\|\Phi_{k,\zeta}\|_{L^2(\Omega(\zeta))}=1$ $(\Phi_{k,\zeta}\cdot\Phi_{m,\zeta})_{L^2(\Omega(\zeta))}=0$, $k\neq m$) which are separated as follows $\{\Phi_{k,\zeta}\}_{k=1}^{\infty}=\{\phi_{k,\zeta}\}_{k=1}^{\infty}\cup\{\psi_{k,\zeta}\}_{k=1}^{\infty}$ where $\Phi_{k,\zeta}$, $\phi_{k,\zeta}$ and $\psi_{k,\zeta}$, respectively correspond to $\mu_k(\zeta)$, $\omega_k(\zeta)$ and $\lambda_k(\zeta)$, and satisfy the following conditions,*

(2.1) $$\lim_{\zeta\to 0}\|\psi_{k,\zeta}\|_{L^2(D_1\cup D_2)}=0,$$

(2.2) $$\lim_{\zeta\to 0}\sup_{x=(x_1,x')\in Q(\zeta)}|d^{1/2}\zeta^{(n-1)/2}\psi_{k,\zeta}(x_1,x')-S_k(x_1)|=0,$$

(2.3) $$\lim_{\zeta\to 0}\sup_{x\in D_1\cup D_2}|\phi_{k,\zeta}(x)-\phi_k(x)|=0,$$

(2.4) $$\lim_{\zeta\to 0}\sup_{x=(x_1,x')\in Q(\zeta)}|\phi_{k,\zeta}(x_1,x')-V_k(x_1)|=0.$$

Here we denoted by V_k the unique solution of the following two point boundary value problem (2.5) *for each $k=1,2,3,\cdots$,*

(2.5) $$\begin{cases}\dfrac{d^2V}{dz^2}+\omega_k V=0, & -1<z<1,\\ V(1)=\phi_k(p_1), & V(-1)=\phi_k(p_2).\end{cases}$$

and d is the $(n-1)$-dimensional Lebesgue measure of the unit ball in $\boldsymbol{R}^{n-1}$.

§ 3. Short argument about the proof.

Let $\{\Phi_{k,\zeta}\}_{k=1}^{\infty}$ be any complete orthonormal system of eigenfunctions in $L^2(\Omega(\zeta))$. We separate it into two families $\{\Phi_{k,\zeta}^{(i)}\}_{k=1}^{\infty}$ $(i=1,2)$ by the following conditions:

(3.1) $$\begin{cases}\overline{\lim_{\zeta\to 0}}\|\Phi_{k,\zeta}^{(1)}\|_{L^\infty(\Omega(\zeta))}=\infty,\\ \overline{\lim_{\zeta\to 0}}\|\Phi_{k,\zeta}^{(2)}\|_{L^\infty(\Omega(\zeta))}<\infty.\end{cases}$$

It is easy to see,

$$\lim_{\zeta\to 0}\|\Phi_{k,\zeta}^{(2)}\|_{L^2(D_1\cup D_2)}=1,\qquad \underline{\lim_{\zeta\to 0}}\|\bar{\Phi}_{k,\zeta}\|_{L^2(D_1\cup D_2)}=0.$$

where $\bar{\Phi}_{k,\zeta}(x)=\Phi_{k,\zeta}^{(1)}(x)/\|\Phi_{k,\zeta}^{(1)}\|_{L^\infty(\Omega(\zeta))}$.

The important point in our proof is to characterize the behavior of the eigenfunction in the sense of the uniform convergence. We can apply the method developed in [2] and [3] to $\{\bar{\Phi}_{k,\zeta}\}_{k=1}^{\infty}$ and $\{\Phi_{k,\zeta}^{(2)}\}_{k=1}^{\infty}$ which are uniformly bounded. Thus we can prove that each $\bar{\Phi}_{k,\zeta}$ converges uniformly to 0 in $D_1\cup D_2$ and any sequence of positive values $\{\zeta_m\}\downarrow 0$ has a subsequence $\{\sigma_m\}$ such that $\bar{\Phi}_{k,\sigma_m}$ converges to S_k or $-S_k$ when $m\to\infty$ in $Q(\sigma_m)$. Similarly, we can deduce the asymptotic behavior of $\Phi_{k,\zeta}^{(2)}$.

References

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[3] ——: Singular perturbation of domains and semilinear elliptic equation. II (preprint).
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