76. A Note on p-adic Etale Cohomology

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1. Let X be a projective smooth scheme over a complete discrete valuation ring A of mixed characteristics $(0, p)$. In [2], Fontaine and Messing studied the relation between the p-adic etale cohomology of the generic fiber $H^*_{et}(X_{\bar{\eta}})=H^*_{et}(X_\eta\otimes\bar{\eta}, Z_p)$ ($\bar{\eta}$ is an algebraic closure of η) and the crystalline cohomology of the special fiber $H^*_{crys}(X_s)$. In this article, we consider not $Gal(\bar{\eta}/\eta)$-representation $H^*_{et}(X_{\bar{\eta}})$, but $H^*_{et}(X_\eta)$ itself and study this cohomology group by using the syntomic cohomology introduced in [2]. Detailed studies containing the complete proof will appear elsewhere.

We will use the following notation: X is a projective, smooth and geometrically connected scheme over A of dimension d as above, and $Y=X_s$ (resp. X_η) is the special fiber (resp. the generic fiber), and $i: Y\to X$ (resp. $j: X_\eta\to X$) is the canonical morphism. We assume that the residue field F of A has a finite p-base of order g (i.e. $[F:F^p]=p^g$).

Fontaine and Messing [2] defined the syntomic site X_{syn} and a sheaf S_n^r on X_{syn} in order to link the etale cohomology to De Rham cohomology. This sheaf S_n^r is regarded as an "ideal" etale sheaf $Z/p^n(r)$ on X. Namely, the group $H^q(X_{syn}, S_n^r)$ is expected to play a role of "$H^q(X_{et}, Z/p^n(r))$" which cannot be defined directly. In [2], a global cohomology $H^q(X_{\bar{\eta}}, Z_p)$ was studied under the assumption $e_A=\mathrm{ord}_A(p)=1$. Our aim in this paper is a local study of p-adic etale vanishing cycles $i^*Rj_*Z/p^n(r)$ when e_A may not be 1. Put $\mathcal{S}_n(r)=i^*R\pi_*S_n^r\in D(Y_{et})$ as in [3] where $\pi: X_{syn}\to X_{et}$ is the canonical morphism. Fontaine and Messing defined a morphism $S_n^r\to i'^*j'_*Z/p^n(r)$ (where $j': X_{\eta et}\to X_{syn-et}$, $i': X_{syn}\to X_{syn-et}$) in [2] 5, which induces $\mathcal{S}_n(r)\to i^*Rj_*Z/p^n(r)$. We study the difference between $\mathcal{S}_n(r)$ and $i^*Rj_*Z/p^n(r)$.

Theorem. *If $r<p-1$, there exists a distinguished triangle*

$$\mathcal{S}_n(r)\longrightarrow\tau_{\leq r}i^*Rj_*Z/p^n(r)\longrightarrow W_n\Omega^{r-1}_{Y\,lo}[-r].$$

where $W_n\Omega^{r-1}_{Y\,log}$ is the logarithmic Hodge-Witt sheaf. In particular, if $r\geq d(=\dim X)+g(=\mathrm{ord}_p[F:F^p])$, we have a long exact sequence

$$\begin{aligned}&\longrightarrow H^q(X_{syn}, S_n^r)\longrightarrow H^q(X_{\eta\,et}, Z/p^n(r))\longrightarrow H^{q-r}(Y_{et}, W_n\Omega^{r-1}_{Y\,log})\longrightarrow\\ &H^{q+1}(X_{syn}, S_n^r)\longrightarrow H^{q+1}(X_{\eta\,et}, Z/p^n(r))\longrightarrow H^{q-r+1}(Y_{et}, W_n\Omega^{r-1}_{Y\,log})\longrightarrow.\end{aligned}$$

In the case $e_A=ord_A(p)=1$ and $r\geq d+g$, considering

$$\mathcal{S}_n(r)\simeq DR(X\otimes Z/p^n)[-1]$$

($DR(T)$ means the De Rham complex $\Omega^\cdot_{T/Z}$), we have

Corollary 1. *Suppose that $e_A=ord_A(p)=1$ and $d+g\leq r<p-1$. Then,*

we have a long exact sequence

$$\cdots\longrightarrow H^{q-1}_{DR}(X\otimes Z/p^n)\longrightarrow H^q(X_{\eta\,et}, Z/p^n(r))\longrightarrow H^{q-r}(Y_{et}, W_n\Omega^{r-1}_{Y\,log})\longrightarrow\cdots.$$

Corollary 2. *Suppose that $e_A=1$ and the residue field F of A is finite and $d<p-1$. Then, we have a long exact sequence*

$$\cdots\longrightarrow H^q(Y_{et}, Z/p^n)\longrightarrow H^q(X_{\eta\,et}, Z/p^n)\longrightarrow H^{q-1}_{crys}(Y/W_n)\longrightarrow\cdots.$$

where $W_n=W_n(F)$ and $H^*_{crys}(Y/W_n)$ is the crystalline cohomology of Y.

This can be seen from Corollary 1 by considering the duality. This Corollary 2 gives another proof of the following result [4] Prop. 7 in the case $e_A=1$.

Corollary 3. *Every abelian etale covering of X_η comes from some abelian etale covering of Y and some abelian extension of η.*

This follows from corollary 2 immediately. In fact, since $H^2(Spec\, F_{et}, Z/p^n)=0$, we have a diagram of exact sequences

$$\begin{array}{ccccccc}
0\longrightarrow & H^1(Y_{et}, Z/p^n) & \longrightarrow & H^1(X_{\eta\,et}, Z/p^n) & \longrightarrow & H^0_{crys}(Y) & \\
& \uparrow & & \uparrow & & \uparrow & \\
0\longrightarrow & H^1_{et}(Spec\, F, Z/p^n) & \longrightarrow & H^1_{et}(\eta, Z/p^n) & \longrightarrow & H^0_{crys}(F/W_n)=W_n & \longrightarrow 0.
\end{array}$$

Therefore, the following is surjective. $H^1(Y, Z/p^n)\oplus H^1(\eta, Z/p^n)\to H^1(X_\eta, Z/p^n)$. Q.E.D.

Remark. The author gave an explicit definition of the homomorphism $H^1_{et}(\eta, Z/p^n)\to W_n(F)$ in a general situation (F is arbitrary) [5] for a henselian discrete valuation field η with $e_A=1$.

2. We review the description of $\mathcal{S}_n(r)$ in [3]. We take a complete discrete valuation ring $A_0\subset A$ such that $e_{A_0}=1$ and the residue field of A_0 is isomorphic to F, $A_0/p\xrightarrow{\sim}F$. (The existence of such a ring follows from [0] IX §2 Th. 1.) Furthermore, take a closed immersion $X\to Z$ over A_0 where Z is smooth over A_0 and has a Frobenius endomorphism f, which means f mod p is the absolute Frobenius of $Z\otimes Z/p$. Denote $X_n=X\otimes Z/p^n$ and $Z_n=Z\otimes Z/p^n$ for $n\geq 1$, and let $D_n=D_{X_n}(Z_n)$ be the PD. envelope and J_{D_n} be the ideal of D_n corresponding to X_n, and $J^{[r]}_{D_n}$ its r-th divided power for $r\geq 1$. For $r\leq 0$, $J^{[r]}_{D_n}$ is defined to be $\mathcal{O}_{D_n}$. We define $\mathcal{J}^{[r]}_{D_n}$ by the complex of sheaves on Y_{et};

$$J^{[r]}_{D_n}\longrightarrow J^{[r-1]}_{D_n}\otimes_{\mathcal{O}_{D_n}}\Omega^1_{Z_n}\longrightarrow J^{[r-2]}_{D_n}\otimes_{\mathcal{O}_{D_n}}\Omega^2_{Z_n}\longrightarrow\cdots.$$

Assume $r<p-1$. For a Frobenius morphism f of Z, $\mathrm{f}_r: \mathcal{J}^{[r]}_{D_n}\to\mathcal{J}^{[0]}_{D_n}$ is defined by "p^{-r}f". Then, the complex $\mathcal{S}_n(r)$ is isomorphic to the mapping fiber of $\mathrm{f}_r-1: \mathcal{J}^{[r]}_{D_n}\to\mathcal{J}^{[0]}_{D_n}$. Explicitly, $\mathcal{S}_n(r)$ is as follows.

(2.1) $$\cdots\longrightarrow(J^{[r-i]}_{D_n}\otimes\Omega^i_{Z_n})\oplus(\mathcal{O}_{D_n}\otimes\Omega^{i-1}_{Z_n})\longrightarrow\cdots$$
$$(x, y)\longmapsto(dx, (\mathrm{f}_r-1)(x)-dy).$$

Note that this complex is independent of the choice of Z and f in $D(Y_{et})$.

3. For the proof of Theorem, since $\mathcal{H}^q(\mathcal{S}_n(r))=0$ for $q>r$, it suffices to show that $\mathcal{S}_n(r)\to i^*Rj_*Z/p^n(r)$ induces an isomorphism

(3.1) $$\mathcal{H}^q(\mathcal{S}_n(r))\xrightarrow{\sim}i^*R^qj_*Z/p^n(r)\qquad\text{if } q<r<p-1$$

and an exact sequence

(3.2) $$0\longrightarrow\mathcal{H}^q(\mathcal{S}_n(q))\longrightarrow i^*R^qj_*Z/p^n(q)\longrightarrow W_n\Omega^{q-1}_{Y\,log}\longrightarrow 0.$$

Put $M_n^q = i^* R^q j_* Z/p^n(q)$ and denote by $U^0 M_n^q$ the subsheaf of M_n^q generated locally by $\{a_1, \cdots, a_q\}$ with $a_1, \cdots, a_q \in i^* \mathcal{O}_X^*$ where $\{a_1, \cdots, a_q\}$ means the "symbol" ([1] § 1). In [1], an exact sequence $0 \to U^0 M_n^q \to M_n^q \to W_n \Omega_{Y\,log}^{q-1} \to 0$ was obtained. The exact sequence (3.2) is a consequence of an isomorphism

(3.3) $$\mathcal{H}^q(\mathcal{S}_n(q)) \xrightarrow{\sim} U^0 M_n^q.$$

In order to prove (3.1) and (3.3), by a standard argument, we may assume $n=1$.

4. In this section, in order to prove (3.1) and (3.3), we study the structure of $\mathcal{H}^q(\mathcal{S}_1(r))$ for $q \leq r < p-1$. Our aim is to define some complexes $gr^i \mathcal{S}_1(r)$ whose cohomology groups $\mathcal{H}^q(gr^i \mathcal{S}_1(r))$ give subquotients of $\mathcal{H}^q(\mathcal{S}_1(r))$ and to compute these cohomology groups $\mathcal{H}^q(gr^i \mathcal{S}_1(r))$.

Since our problem to prove (3.1) and (3.3) is local, we may assume X is a projective space $\boldsymbol{P}_A^m$. In the following, we will use the explicit description of $\mathcal{S}_1(r)$ (2.1) and the same notation as in 2. Take A_0 such that $e_{A_0}=1$ and A/A_0 is totally ramified and take a prime element π of A. Let $f(T) \in A_0[T]$ be the monic minimal polynomial of π over A_0. Take $Z = \boldsymbol{P}_{A_0}^m[T]$ and define a closed immersion $X \to Z$ by $f(T)$, and define a Frobenius f of Z such that $\mathrm{f}(T) = T^p$. A filtration of $\mathcal{H}^q(\mathcal{S}_1(r))$ is defined by using these Z and T. We need some more notation. For $h \in \boldsymbol{Q}$ and an ideal I of $\mathcal{O}_{D_1}$, $T^h I$ is an ideal generated by $T^m I$ such that $m \geq h$ and $m \in \boldsymbol{N}$. For $i \in \boldsymbol{N}$ and $s \in \boldsymbol{Z}$, an ideal $J_i^{[s]}$ of $\mathcal{O}_{D_1}$ is defined by $J_i^{[s]} = (T^i \mathcal{O}_{D_1} + J_{D_1}^{[p]}) \cap (T^{(ip-1)} J_{D_1}^{[s]} + J_{D_1}^{[p]})$. For an ideal I of $\mathcal{O}_{D_1}$, $I \otimes (\Omega_{Z_1}^q)'$ is the subsheaf of $\mathcal{O}_{D_1} \otimes \Omega_{Z_1}^q$ generated by $I \otimes \Omega_{Z_1}^q$ and the elements of the form $a \cdot dlog\, T$ with $a \in I \otimes \Omega_{Z_1}^{q-1}$.

For $i \geq 0$, a complex $U^i \mathcal{J}_{D_1}^{[r]}$ is defined as follows.

(4.1) $$U^i \mathcal{J}_{D_1}^{[r]} : J_i^{[r]} \longrightarrow J_i^{[r-1]} \otimes (\Omega_{Z_1}^1)' \longrightarrow J_i^{[r-2]} \otimes (\Omega_{Z_1}^2)' \longrightarrow \cdots$$

As in the case $\mathcal{J}_{D_n}^{[r]}$, we can define $\mathrm{f}_r = \text{“}p^{-r}\mathrm{f}\text{”} : U^i \mathcal{J}_{D_1}^{[r]} \longrightarrow U^i \mathcal{J}_{D_1}^{[0]}$ for $r < p-1$. Moreover, we define $gr^i \mathcal{J}_{D_1}^{[r]}$ by an exact sequence

$$0 \longrightarrow U^{i+1} \mathcal{J}_{D_1}^{[r]} \longrightarrow U^i \mathcal{J}_{D_1}^{[r]} \longrightarrow gr^i \mathcal{J}_{D_1}^{[r]} \longrightarrow 0.$$

Then, $U^i \mathcal{S}_1(r)$ (resp. $gr^i \mathcal{S}_1(r)$) is defined to be the mapping fiber of $\mathrm{f}_r - 1 : U^i \mathcal{J}_{D_1}^{[r]} \longrightarrow U^i \mathcal{J}_{D_1}^{[0]}$ (resp. $\mathrm{f}_r - 1 : gr^i \mathcal{J}_{D_1}^{[r]} \longrightarrow gr^i \mathcal{J}_{D_1}^{[0]}$).

The following can be seen by an explicit calculation.

Lemma (4.2). *For $i \geq 0$ and $q \geq 0$, $\mathcal{H}^q(U^{i+1} \mathcal{S}_1(r)) \to \mathcal{H}^q(U^i \mathcal{S}_1(r))$ is injective.*

By this lemma, we can regard $\mathcal{H}^q(U^i \mathcal{S}_1(r))$ as a filtration of $\mathcal{H}^q(\mathcal{S}_1(r))$. Put $L_1^q(r) = \mathcal{H}^q(\mathcal{S}_1(r))$, $U^i L_1^q(r) = \mathcal{H}^q(U^i \mathcal{S}_1(r))$, and $gr^i L_1^q(r) = U^i L_1^q(r) / U^{i+1} L_1^q(r)$. We shall calculate $gr^i L_1^q(r)$. By Lemma (4.2), we have $gr^i L_1^q(r) = \mathcal{H}^q(gr^i \mathcal{S}_1(r))$.

Proposition (4.3). *Suppose $0 \leq q \leq r < p-1$ and $i \geq 0$, and put $e = e_A$.*

1) *If $i < ep(r-q)/(p-1)$ or $i \geq ep(r-q+1)/(p-1)$, $gr^i L_1^q(r) = 0$.*

2) *The case $i = ep(r-q)/(p-1)$. (This case only occurs when $e(r-q)$ is divisible by $p-1$.)*

$$gr^i L_1^q(r) = \begin{cases} \Omega_{Y\,log}^q & \text{if } q = r \\ \Omega_{Y\,log}^q \oplus \Omega_{Y\,log}^{q-1} & \text{if } q < r \end{cases}$$

3) *Assume $ep(r-q)/(p-1) < i < ep(r-q+1)/(p-1)$.*

i) *If i is not divisible by p,* $gr^i L_1^q(r) = \Omega_Y^{q-1}$.

ii) *If i is divisible by p,* $gr^i L_1^q(r) = B\Omega_Y^q \oplus B\Omega_Y^{q-1}$ *where* $B\Omega_Y^j = Image(d: \Omega_Y^{j-1} \to \Omega_Y^j)$.

On the other hand, the structure of $gr^i M_1^q$ is determined in [1] Cor. (1.4.1). The isomorphism (3.3) $L_1^q(q) \rightleftarrows U^0 M_1^q$ is verified by comparing $gr^i L_1^q(q)$ with $gr^i M_1^q$. (The compatibility of the symbol maps from Milnor K-sheaf to $L_1^q(q)$ ([3] I §3) and to M_1^q ([1] §1) shows that the map $L_1^q(q) \to M_1^q$ induces $gr^i L_1^q(q) \to gr^i M_1^q$.)

Next, we show (3.1). Let ζ_p be a primitive p-th root of unity, $G = Gal(A[\zeta_p]/A)$ be the Galois group of $A[\zeta_p]/A$, and $\overline{M}_1^q$ be the sheaf obtained by the base change $Spec\, A[\zeta_p] \to Spec\, A$. We have to show the bijectivity of

$$L_1^q(r) \xrightarrow{\sim} i^* R^q j_* Z/p(r) \simeq \overline{M}_1^q(r-q)^G.$$

This is also proved by comparing the filtrations using Prop. (4.3) and the structure theorem on $\overline{M}_1^q$ in [1]. (The above induces $U^i L_1^q(r) \to U^{ih - e'(r-q)h} \overline{M}_1^q$ where $h = \sharp G$ and $e' = e_A p/(p-1)$.)

Remark. The definition of *gr*-complex was suggested by K. Kato. The author gave a different proof of Theorem in his master's thesis, which uses a relation between Milnor K-groups and differential modules.

References

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