74. On Subcommutative Rings

By G. W. S. van ROOYEN
Mathematics Department, University of Stellenbosch

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Introduction. R denotes an associative ring, not necessarily with identity. In [1] Barbilian defines R to be subcommutative if $Rx \subseteq xR$ for all $x \in R$, and in [6] Reid defines R to be subcommutative if $xR \subseteq Rx$ for all $x \in R$. The case $xR = Rx$ for all $x \in R$ implies R is a duo ring i.e. every one-sided ideal of R is a two-sided ideal (see [7]). Whether one prefers the concept of *left* subcommutativity (Reid) or the concept of *right* subcommutativity (Barbilian) seems to be really immaterial. For on the one hand, theorems may be proved from the side preferred and they follow by symmetry from the other; and on the other hand R is right subcommutative iff the opposite ring of R is left subcommutative. In this paper we examine connections between subcommutativity and related concepts in both the unital and non-unital cases. The results are somewhat scattered, but they touch upon several interesting classes of rings. *Subcommutative* will mean right subcommutative, and the word *ideal* without modifier will mean two-sided ideal. We will work on the right.

Subcommutativity and reflexivity. We require concepts of the following kind: Call a right ideal I of R *reflexive* [5] if $xRy \subseteq I$ implies $yRx \subseteq I$ where $x, y \in R$, and assign the term *completely reflexive* [5] to those I for which $xy \in I$ implies $yx \in I$.

Definition. *A right ideal I of R is called quasi-reflexive if whenever X and Y are right ideals of R with $XY \subseteq I$ then $YX \subseteq I$.*

One easily sees that a quasi-reflexive ideal is two-sided. In the unital case the concepts of reflexivity and quasi-reflexivity coincide [5, proposition 2.3]. Complete reflexivity implies quasi-reflexivity. We write $(a)_r$ for the principal right ideal generated by $a \in R$. Then standard arguments yield the following

Lemma. *A right ideal I of R is quasi-reflexive iff $(x)_r(y)_r \subseteq I$ implies $(y)_r(x)_r \subseteq I$ where $x, y \in R$.*

Any prime (semi-prime) ideal of R is quasi-reflexive. Hence the intersection of any set of prime (semi-prime) ideals is quasi-reflexive. This implies that any ideal in a (von Neumann) regular ring is quasi-reflexive. We also note the following: R subcommutative implies R is right duo (i.e. every right ideal of R is two-sided), consequently $(a) = (a)_r$. This fact establishes one part of

Proposition 1. *Let R be subcommutative. Then an ideal I of R is completely reflexive iff it is quasi-reflexive. Moreover, the subset of nil-*

potent elements in R forms a completely reflexive ideal.

Proof. Consider $xy \in I$; $x, y \in R$. Straightforward calculations show the ideal $(x)(y)$ is contained in I since R is subcommutative. Suppose I is quasi-reflexive. By the above Lemma, $(y)(x) \subseteq I$ whence $yx \in I$, and so I is completely reflexive. The second assertion follows easily.

The foregoing proposition and [5, Proposition 2.3] lead at once to

Corollary 1. *Let I be an ideal of a subcommutative ring with identity. The following are equivalent.*

a) *I is quasi-reflexive.*

b) *I is completely reflexive.*

c) *I is reflexive.*

Following Mason [5] we call right ideal I right (left) symmetric if $abc \in I$ implies $bac \in I$ $(acb \in I)$ where $a, b, c \in R$. We give now ideal-theoretical characterizations of a special class of subcommutative rings in both the unital and non-unital cases in improving [5, Theorem 3.1(a) and Corollary (a) p. 1719].

Proposition 2. *The following are equivalent for the ring R.*

a) *Every right ideal of R is completely reflexive.*

b) *Every right ideal of R is quasi-reflexive.*

c) *$AB = BA$ whenever A and B are right ideals of R.*

d) *$(x)_r(y)_r = (y)_r(x)_r$ for all elements x and y in R.*

e) *The equation $xy = ys$ always has a solution s in $(x)_r$, given $x, y \in R$.*

f) *Every principal right ideal of R is completely reflexive.*

g) *Every principal right ideal of R is quasi-reflexive.*

If in addition R has identity 1, then these are equivalent to:

h) *Every right ideal of R is reflexive.*

i) *Every principal right ideal of R is reflexive.*

j) *Every right ideal is left and right symmetric.*

k) *Every principal right ideal is left and right symmetric.*

l) *$xyR = yxR$ for all $x, y \in R$, i.e. R is right interversive* [5].

Proof. Clearly a)$\Rightarrow$b)$\Rightarrow$c)$\Rightarrow$d). d)$\Rightarrow$e) since $xy \in (y)_r(x)_r$. e)$\Rightarrow$f): For let $xy \in (t)_r$, $t \in R$. But for some $v \in R$ and integer m, $yx = x(my + yv)$ implies $yx \in (t)_r$. f)$\Rightarrow$g) follows immediately. g)$\Rightarrow$a): Let $xy \in I$. Now $(x)(y) \subseteq (xy)$ since $(y) = (y)_r$ and $(x) = (x)_r$. Consequently $yx \in (xy)$ and so $yx \in I$. Thus a)$\sim$g) are equivalent.

In the unital case b)$\Rightarrow$h), g)$\Rightarrow$i) follow from Corollary 1. Therefore a)$\sim$i) are equivalent. To prove h)$\Rightarrow$j), note that R is subcommutative. Corollary 1 implies that every right ideal is completely reflexive, so we can apply e). Thus $yx = xyt$. For any $r \in R$, $yxr = xy(tr) = xyrs$ where $r \in R$. So, if $xyr \in I$ then $yxr \in I$. This proves I is right symmetric. Straightforward calculations show I is also left symmetric. j)$\Rightarrow$k), k)$\Rightarrow$l) follow easily. Finally l)$\Rightarrow$h). Let $xRy \subseteq I$. In $r \in R$, $yrx = yxrs = xytrs = xtrsyu \in xRyu \subseteq I$. Therefore $yRx \subseteq I$.

Definition. *We call a ring strongly subcommutative if it satisfies* *conditions* a)–g).

A direct consequence of e) of the preceding proposition is

Corollary 2. *Any ideal of a strongly subcommutative ring is subcommutative.*

Every division ring is strongly subcommutative. Moreover, any regular, right duo ring R is strongly subcommutative since every (right) ideal is semi-prime and hence quasi-reflexive. In particular R is strongly regular. For, if $x \in R$ then $x = xyx = x^2r$; $y, r \in R$. It is well known that any strongly regular ring is a regular duo ring [4]. So we conclude a familiar result, i.e. a ring is regular, right duo iff it is strongly regular [2, cf. criteria (1) and (5)].

It is well known that every idempotent of a strongly regular ring is central. We prove

Proposition 3. *In a strongly subcommutative ring idempotents are central.*

Proof. Suppose R is strongly subcommutative, e an idempotent of R. Then $(se-s)e=0$ for all $s \in R$. Apply how part b) of the preceding proposition, $e(se-s)=0$ and so $ese=es$. Likewise $ese=se$. Therefore $es=se$ for all $s \in R$. □

Let $x \in R$. Denote the centralizer of x in R by the symbol $C(x)$.

Corollary 3. *Let R be strongly subcommutative and suppose that for some $x \in R$, $C(x)$ has an identity e. Then e is the identity for R.*

Proof. If $a \in R$, define $y=a-ea$. Since e is central idempotent $yx=0$ and $xy=0$, so $y \in C(x)$ whence $y=ye=0$. Therefore $a=ea=ae$ for all $a \in R$. □

Remark. The preceding corollary was originally proved by Herstein-Neumann in the case for semi-prime rings [3, Lemma 1].

Proposition 4. *Let R be a strongly subcommutative ring, e a non-zero idempotent of R. Then*

a) *eR is a strongly subcommutative ring with identity e.*

b) *eR is a minimal (right) ideal of R iff eR is a division ring.*

c) *If eR is minimal and the right (left) annihilator* ann(e) *of e is zero, then R is a division ring.*

Proof. a) The ideal eR of R is right duo (Corollary 2). Let I be any ideal of eR and x be any element of I. Let r be any element of R. Then $xr=exr=xer \in I$. Thus I is an ideal of R. b) Assume eR is a minimal (right) ideal of R. Clearly e is the identity of eR. If $0 \neq x \in eR$ then $xeR = eR$ and $e=xy$, $y \in eR$. Apply Corollary 2 on eR. There exists an element u of eR such that $xy=yu$ whence $x=u$. This proves x is invertible. The converse statement is obvious. c) Assume eR is minimal and ann$(e)=0$. Consequently, the ideal $\{x-ex \mid x \in R\}$ vanishes. Thus $eR=R$. Part b) completes the proof. □

Corollary 4. *Every non-nilpotent minimal ideal I of a strongly subcommutative ring is a division ring* [7, Proposition 2].

Corollary 5. *If R is a strongly subcommutative, subdirectly irreducible non-zero ring without non-zero nilpotent elements then R is a division ring* [7, Theorem 1].

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