70. Some Structure Theorems for ω-Stable Groups

By Katsumi TANAKA

The Graduate School of Science and Technology, Kobe University

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 14, 1987)

§ 1. Introduction. An ω-stable group is a structure G whose complete theory in the first order language of G, $L(G)$, is ω-stable and the group operation on G is 0-definable in $L(G)$. In this paper we study some structure theorems for ω-stable groups. One of the earliest results in this subject shown by Baldwin and Saxl [1] is that a locally nilpotent (ω-)stable group is solvable. The point of the proof is that every (ω-)stable group G satisfies the minimal condition on centralizers, i.e., G is an $\mathcal{M}_c$-group.

In a previous paper [7] we studied local properties of ω-stable groups of finite Morley rank. In that article we showed that a locally solvable (locally nilpotent) ω-stable group of finite Morley rank is solvable (nilpotent-by-finite). Recently we realized that these results hold for CZ-groups satisfying the maximal condition on closed connected subgroups. A CZ-group is a group G which carries a T_1-topology satisfying the minimal condition on closed sets, such that for each $a \in G$ the following maps from G to itself are continuous:

$$x \mapsto xa, \quad x \mapsto ax, \quad x \mapsto x^{-1}, \quad x \mapsto x^{-1}ax.$$

It is known that every CZ-group is an $\mathcal{M}_c$-group. CZ-groups were introduced by Kaplansky [5] as an abstraction of linear groups. On the other hand, it is well-known that ω-stable groups of finite Morley rank are quite similar to linear groups over algebraically closed fields. Hence it is reasonable that these two classes of groups share the same structure theorems.

In [4], Higgins proved that a locally supersolvable CZ-group satisfying the maximal condition on closed connected subgroups is nilpotent-by-finite, and hence hypercyclic. Our result is the same vein.

Theorem. *A locally supersolvable ω-stable group of finite Morley rank is nilpotent-by-finite, and hence hypercyclic.*

Most of the proof of the original theorem for CZ-groups go through in our context. Hence our main interest is not the particular results which are proved but some behaviors of model closures in ω-stable groups.

The author is profoundly grateful to Professor John T. Baldwin for his helpful suggestions.

§ 2. Model closures. Our notations are standard. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of groups. A group is said to be $\mathcal{A}$-by-$\mathcal{B}$ if G has a normal subgroup N such that $N \in \mathcal{A}$ and $G/N \in \mathcal{B}$. A group G is locally $\mathcal{A}$ if every finitely generated subgroup of G is in $\mathcal{A}$. In this note "definable" always means definable with parameters.

Let G be an ω-stable group and X a subset of G. Let $\{H_i\}_{i \in I}$ be a collection of all definable subgroups of G including X. Define the *model closure* of X, $\tilde{X}$, by $\bigcap_{i \in I} H_i$. Then by the ω-stable descending chain condition, there is a finite subset J of I such that $\bigcap_{i \in I} H_i = \bigcap_{i \in J} H_i$. Hence $\tilde{X}$ is a definable subgroup of G. Obviously, every subset of G has a unique model closure. Denote by G° the connected component of G. For the elementary properties of ω-stable groups see [3]. The following are our key lemmas.

Lemma 2.1. *Let G be an ω-stable group, and H and K subgroups of G. If H is a normal subgroup of K, then $\tilde{H}$ is a normal subgroup of $\tilde{K}$.*

Proof. For each element a in K, we have $H^a = H$ and they are included in $\tilde{H}^a$. Since $\tilde{H}^a$ is definable, so is $\bigcap_{a \in K} \tilde{H}^a$ by the ω-stable descending chain condition. Hence H is included in $\bigcap_{a \in K} \tilde{H}^a$. By the minimality of $\tilde{H}$ we have $\tilde{H} = \bigcap_{a \in K} \tilde{H}^a$, i.e., the group K normalizes $\tilde{H}$. Since $N_G(\tilde{H})$ is definable including K, it follows from the minimality of $\tilde{K}$ that $\tilde{K}$ is included in $N_G(\tilde{H})$. Hence $\tilde{H}$ is a normal subgroup of $\tilde{K}$.

Lemma 2.2. *Let A, B and C be subgroups of an ω-stable group G. If C is a normal subgroup of G and $[A, B] \subseteq C$, then $[\tilde{A}, \tilde{B}] \subseteq \tilde{C}$.*

Proof. Fix an element b in B. Let $X_b = \{a \in \tilde{A} : [a, b] \in \tilde{C}\}$. Then it is easy to check that X_b is a definable subgroup. Since $[A, B] \subseteq C$, we have $A \subseteq X_b \subseteq \tilde{A}$. Hence $X_b = \tilde{A}$. So $[\tilde{A}, B] \subseteq \tilde{C}$. Fix an element a in $\tilde{A}$. Let $Y_a = \{b \in \tilde{B} : [a, b] \in \tilde{C}\}$. Then Y_a is also a definable subgroup. Since $[\tilde{A}, B] \subseteq \tilde{C}$, it follows that $B \subseteq Y_a \subseteq \tilde{B}$. Hence $Y_a = \tilde{B}$, so $[\tilde{A}, \tilde{B}] \subseteq \tilde{C}$.

In Lemma 2.2, moreover if G has finite Morley rank and if $[A, B] = C$ then $[\tilde{A}, \tilde{B}] = \tilde{C}$. This is a consequence of Zil'ber's indecomposability theorem. Also the superstable version can be found in [2]. The next lemma follows from Lemma 2.2, and this generalizes [9; Lemma 1].

Lemma 2.3. *Let G be an ω-stable group, H a subgroup of G. Then*

(i) *H is solvable of derived length d if and only if $\tilde{H}$ is.*

(ii) *H is nilpotent of class c if and only if $\tilde{H}$ is.*

§3. Structure theorems. The following lemma is convenient.

Lemma 3.1. *Let G be an ω-stable group and $N \lhd M$ subgroups of G. If M/N is finite, then so is $\tilde{M}/\tilde{N}$.*

Proof. By Lemma 2.1, $\tilde{N}$ is a normal subgroup of $\tilde{M}$. Let $M = \langle a_1, \cdots, a_n \rangle N$. Then $\langle a_1, \cdots, a_n \rangle \tilde{N}$ is definable and includes M, so $|\tilde{M}/\tilde{N}| \leq |M/N|$.

Definition. A group G is said to be *supersolvable* if G has a normal series whose factors are cyclic.

Lemma 3.2. *A locally supersolvable ω-stable group of finite Morley rank is "locally nilpotent"-by-"locally finite abelian".*

Proof. Let G be a locally supersolvable ω-stable group of finite Morley rank. Since G has finite Morley rank, we can choose a finitely generated subgroup X of G such that $\tilde{X}^\circ$ is a maximal member of the collection of all $\tilde{Y}^\circ$ with Y a finitely generated subgroup of G.

Claim 1. $\tilde{X}^\circ \lhd G$.

For all $g \in G$, we have $\langle \widetilde{g, X} \rangle^\circ = \tilde{X}^\circ$ and so $(\tilde{X}^\circ)^g = \tilde{X}^\circ$.

Claim 2. $G/\tilde{X}^\circ$ is locally finite.

Suppose that $Y/\tilde{X}^\circ$ is finitely generated, say

$$Y/\tilde{X}^\circ = \langle y_1\tilde{X}^\circ, \cdots, y_n\tilde{X}^\circ \rangle.$$

Let $Z = \{y_1, \cdots, y_n\} \cup X$. By the maximality of $\tilde{X}^\circ$, we have $\tilde{Z}^\circ = \tilde{X}^\circ$. So $\tilde{Z}/\tilde{X}^\circ$ is finite. Hence $Y/\tilde{X}^\circ$ is finite.

Since every supersolvable group is nilpotent-by-finite, $\tilde{X}^\circ$ is nilpotent by Lemma 2.3 and 3.1.

Claim 3. G' is locally nilpotent.

If S is a finite subset of G' there is a finitely generated subgroup T of G such that $S \subseteq T'$. However, since G is locally supersolvable, T is supersolvable and so T' is nilpotent.

Thus $\tilde{X}^\circ G'$ is locally nilpotent as in [6; 2.31]. Obviously $\tilde{X}^\circ G' \lhd G$ and $G/\tilde{X}^\circ G'$ is abelian. Finally $G/\tilde{X}^\circ G'$ is a homomorphic image of $G/\tilde{X}^\circ$ and so is locally finite.

Theorem. *A locally supersolvable ω-stable group of finite Morley rank is nilpotent-by-finite, and hence hypercyclic.*

Proof. A locally supersolvable group which is nilpotent-by-finite is necessarily hypercyclic (see [8]).

Let G be a counterexample with minimal (Morley rank, degree) in the lexicographic order. Then G is connected. If not, G° is nilpotent-by-finite, and so is G. Without loss of generality we can assume that G is centerless. By Lemma 3.2 the group G' is locally nilpotent and hence by [7; Theorem B], is hypercentral. Thus since $G' \neq 1$, $Z(G') \neq 1$. Since $Z(G) = 1$, we have

$$G \supset C_G(Z(G')) \supseteq G' \neq 1.$$

Since every stable group satisfies the maximal condition on centralizers (the maximal and minimal conditions on centralizers are equivalant for all groups), we may choose C to be a maximal member of the collection of proper centralizers, C^*, in G of abelian subgroup of G such that $C^* \supseteq G'$. From above we see that this collection is non-empty. Since $C \subset G$, we have that C is nilpotent-by-finite. Since $C \supseteq G'$ we have $C \lhd G$, and since C centralizes an abelian subgroup we have $Z(C) \neq 1$.

Consider a finitely generated subgroup of G/C, say for instance $\langle x_1 C, \cdots, x_n C \rangle = \langle x_1, \cdots, x_n \rangle C/C$. Let $\langle x_1, \cdots, x_n \rangle = X$.

Claim. XC is hypercyclic.

Since C is nilpotent-by-finite, so C° is nilpotent. Also, $C^\circ \lhd G$, since $C \lhd G$. Clearly XC/C° is finitely generated so by [8; 11.19] the group XC is hypercyclic.

Now a hypercyclic group has a hypercyclic series including arbitrary normal subgroup. Note that $Z(C) \lhd XC$. It follows in similar way to [4; Theorem C] that $Z(C)$ is neither torsion-free nor torsion. This contradiction completes the proof.

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