68. On Meromorphic and Univalent Functions

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1. Introduction. In the previous paper [1] We derived the area-principle on meromorphic and univalent functions in an annulus and then showed some properties of such functions. In the present paper we shall give the above-mentioned area-principle in the precise form where the omitted area (hereafter defined in Theorem 1) is considered and then improve some results in [1]. Moreover we shall deal with the case of meromorphic and univalent functions in the unit circle $|z|<1$, by means of the results in the case of an annulus.

2. We consider the following annulus

$$D: r<|z|<1 \quad (r>0).$$

Let $w=\varphi_\theta(z, \zeta)$ be regular in D, except for a simple pole of residue 1 at $\zeta \in D$ and univalently map D onto the whole w-plane with two parallel rectilinear slits of the inclination θ. Then $\varphi_\theta(z, \zeta)$ is given as follows ([6], p. 375)

$$\varphi_\theta(z, \zeta)=N(z, \zeta)+e^{i2\theta}M(z, \zeta)$$

where

$$N(z, \zeta)=\frac{1}{z-\zeta}+\frac{1}{\zeta}\sum_{n=1}^{\infty}\frac{r^{2n}((z/\zeta)^{-n}-(z/\zeta)^n)}{1-r^{2n}}=\frac{1}{2}(\varphi_0+\varphi_{\pi/2}).$$

$$M(z, \zeta)=\frac{1}{\bar{\zeta}}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\frac{(z\bar{\zeta})^n}{1-r^{2n}}=\frac{1}{2}(\varphi_0-\varphi_{\pi/2}).$$

We shall give the improved area-principle in the case of an annulus.

Theorem 1. *Let $f(z)$ be regular, except for a simple pole of residue* 1 *at $\zeta \in D$ and univalent in the annulus D. Let δ denote the area of the complementary set of the image domain under $w=f(z)$. (We call δ the omitted area (cf. [4], [7]).) Moreover let $f(z)-N(z, \zeta)=\sum_{n=-\infty}^{\infty} a_n z^n$ in the annulus D. Then we have the following equality.*

$$\sum_{n=-\infty}^{\infty} n(1-r^{2n})|a_n|^2=\pi K(\zeta, \zeta)-\frac{\delta}{\pi},$$

where

$$K(z, \zeta)=\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{n(z\bar{\zeta})^{n-1}}{1-r^{2n}}.$$

denotes the Bergman's kernel function of D.

Proof. We may consider the results in [1] or [5].

Corollary 1. *Let $w=f(z)$ satisfy the same conditions in Theorem* 1 *and δ denote the omitted area of $w=f(z)$. Then we have the following inequality.*

$$|f'(z)-N'(z,\zeta)|\leqq\pi\left(K(\zeta,\zeta)-\frac{\delta}{\pi^2}\right)^{1/2}\cdot K(z,z)^{1/2}.$$

Equality sign holds when $f(z)=\varphi_\theta(z,\zeta)$ *and* $z=\zeta$.

Proof. We can prove the following inequalities by means of Theorem 1 and Schwarz's inequality.

$$\begin{aligned}|f'(z)-N'(z,\zeta)|&=\left|\sum_{n=-\infty}^{\infty}na_nz^{n-1}\right|\leqq\sum_{n=-\infty}^{\infty}(\sqrt{n(1-r^{2n})}|a_n|)(\sqrt{n/(1-r^{2n})}|z|^{n-1})\\&\leqq\left(\sum_{n=-\infty}^{\infty}n(1-r^{2n})|a_n|^2\right)^{1/2}\cdot\left(\sum_{n=-\infty}^{\infty}\frac{n}{1-r^{2n}}|z|^{2n-2}\right)^{1/2}\\&\leqq\left(\pi K(\zeta,\zeta)-\frac{\delta}{\pi}\right)^{1/2}(\pi K(z,z))^{1/2}=\pi\left(K(\zeta,\zeta)-\frac{\delta}{\pi^2}\right)^{1/2}\cdot K(z,z)^{1/2}.\end{aligned}$$

Considering that $M'(\zeta,\zeta)=\pi K(\zeta,\zeta)$ and the omitted area of $w=\varphi_\theta(z,\zeta)$ equals zero, we see that the equality sign holds when $f(z)=\varphi_\theta(z,\zeta)$.

Corollary 2. *Let* $w=f(z)$ *satisfy the same conditions in Theorem* 1 *and* δ *denote the omitted area of* $w=f(z)$. *Then we have the following inequality.*

$$|f(z)-a_0-N(z,\zeta)|\leqq\left(\pi K(\zeta,\zeta)-\frac{\delta}{\pi}\right)^{1/2}A(z)$$

where

$$A(z)=\left(\sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty}\frac{|z|^{2n}}{n(1-r^{2n})}\right)^{1/2}.$$

Next we shall deal with meromorphic and univalent functions in the unit circle $|z|<1$.

Theorem 2. *Let* $w=f(z)$ *be regular, except for a simple pole of residue* 1 *at* $z=\zeta$ $(0<|\zeta|<1)$ *and univalent in the unit circle* $|z|<1$. *Let* δ *denote the omitted area of* $w=f(z)$. *Moreover let*

$$f(z)-\frac{1}{z-\zeta}=\sum_{n=0}^{\infty}b_nz^n(|z|<1).$$

Then we have the following equality which means the area-principle.

$$\sum_{n=1}^{\infty}n|b_n|^2=\frac{1}{(1-|\zeta|^2)^2}-\frac{\delta}{\pi}.$$

Proof. We may make r tend to zero in Theorem 1, considering

$$N(z,\zeta)\to\frac{1}{z-\zeta}\quad\text{and}\quad K(\zeta,\zeta)\to\frac{1}{\pi}\frac{1}{(1-|\zeta|^2)^2}.$$

Remark. This result will generalize Chichra's one ([2], [3]). We can derive the following corollary directly from Theorem 2.

Corollary 3. *Let* $f(z)$ *satisfy the same conditions in Theorem* 2. *Then we have the following inequality.*

$$|b_1|\leqq\frac{1}{1-|\zeta|^2}\sqrt{1-\frac{\delta}{\pi}(1-|\zeta|^2)^2}$$

Equality sign holds when

$$f_0(z)=\beta\frac{z(1-\bar{\zeta}z)}{z-\zeta}\qquad\left(\beta=\frac{1}{\zeta(1-|\zeta|^2)}\right).$$

Remark. With respect to the equality sign, we may consider that $w=f_0(z)$ maps the unit circle $|z|<1$ onto the whole plane with the circular slit and therefore the omitted area equals zero.

By means of the same idea with the proof of Theorem 2, we can derive the following corollary from Corollary 1.

Corollary 4. *Let $f(z)$ satisfy the same conditions in Theorem 2. Then*

$$\left|f'(z)+\frac{1}{(z-\zeta)^2}\right|\leqq\sqrt{\pi}\left(\frac{1}{\pi}\frac{1}{(1-|\zeta|^2)^2}-\frac{\delta}{\pi^2}\right)^{1/2}\frac{1}{1-|z|^2}.$$

Equality sign holds when

$$f_0(z)=\frac{1}{1-|\zeta|^2}\left[\frac{1-\bar{\zeta}z}{z-\zeta}+e^{i2\theta}\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right)\right]$$

(θ is a real arbitrary constant) and $z=\zeta$.

References

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