67. *Mixed Problems for Quasi-Linear Symmetric Hyperbolic Systems*

By Shuichi KAWASHIMA*), Taku YANAGISAWA,**) and Yasushi SHIZUTA**)

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 14, 1987)

1. Introduction. Our primary interest in this note is the mixed problem for the first order quasi-linear hyperbolic systems with characteristic boundary. The case where the boundary matrix is nonsingular has been investigated by several authors, but we do not enter into detail here. (See [5] and the references therein.) The characteristic boundary value problem was treated by Tsuji [6], Majda-Osher [1], Ohkubo [2] and Rauch [4]. Recently, Ohkubo [3] gave an improved version of his sufficient condition for the full regularity of solutions to the linear mixed problem and established a local existence theorem for the quasi-linear mixed problem. Our purpose in this paper is to present another method for solving the quasi-linear mixed problem. To do this, we formulate a new sufficient condition which seems to be somewhat weaker than Ohkubo's one.

2. Assumptions and main result. Let Ω be a bounded domain in $\boldsymbol{R}^n$ with smooth, compact boundary $\partial\Omega$. We study the following mixed problem.

$$(1)_1 \qquad A^0(t,x,u)u_t+\sum_{j=1}^n A^j(t,x,u)u_{x_j}=f(t,x,u) \qquad \text{in } [0,T]\times\Omega,$$

$$(1)_2 \qquad M(x)u=0 \qquad \text{on } [0,T]\times\partial\Omega,$$

$$(1)_3 \qquad u(0,x)=u_0(x) \qquad \text{for } x\in\Omega.$$

Here the unknown $u=u(t,x)$ is a vector-valued function with m components and takes values in a convex open set $\mathcal{O}\subset\boldsymbol{R}^m$, A^0 and A^j, $j=1,\cdots,n$, are smoothly varying real $m\times m$ matrices defined on $[0,T]\times\bar{\Omega}\times\mathcal{O}$, and f is a smooth function on $[0,T]\times\bar{\Omega}\times\mathcal{O}$ with values in $\boldsymbol{R}^m$. M is a real $r\times m$ matrix ($r<m$) depending smoothly on $x\in\partial\Omega$. It is assumed that M is of full rank for $x\in\partial\Omega$.

Condition 1. $A^0(t,x,u)$ is real symmetric and positive definite for $(t,x,u)\in[0,T]\times\bar{\Omega}\times\mathcal{O}$. $A^j(t,x,u)$, $j=1,\cdots,n$, are real symmetric for $(t,x,u)\in[0,T]\times\bar{\Omega}\times\mathcal{O}$.

We write $\partial_j=\partial/\partial x_j$, $j=1,\cdots,n$, and put $\partial_x=(\partial_1,\cdots,\partial_n)$. For a first order differential operator $A(t,x,u;\partial_x)=\sum_{j=1}^n A^j(t,x,u)\partial_j$, we denote its symbol by $A(t,x,u;\xi)=\sum_{j=1}^n A^j(t,x,u)\xi_j$, where $\xi=(\xi_1,\cdots,\xi_n)\in\boldsymbol{R}^n$. Let $\nu(x)$ be the unit outward normal to $\partial\Omega$ at x. The null space of $M(x)$ is the boundary subspace and is denoted by $\mathcal{N}(M(x))$.

---

*) Department of Applied Science, Faculty of Engineering 36, Kyushu University, Fukuoka 812.

**) Department of Mathematics, Nara Women's University, Nara 630.

Condition 2. $\mathcal{N}(M(x))$ is the maximally nonnegative subspace of the boundary matrix $A(t, x, u; \nu(x))$ for $(t, x) \in [0, T] \times \partial\Omega$ and $u \in \mathcal{N}(M(x)) \cap \mathcal{O}$.

We introduce here the notion of the linked system of $(1)_1$. Let us consider a system of the form

$$\tilde{A}^0(t, x, u)w_t + \sum_{j=1}^{n} \tilde{A}^j(t, x, u)w_{x_j} = h(t, x), \tag{2}$$

where $\tilde{A}^0$ and $\tilde{A}^j$, $j=1, \cdots, n$, are smoothly varying real $m \times m$ matrices.

Definition. Suppose that $\tilde{A}^0$ and $\tilde{A}^j$, $j=1, \cdots, n$, satisfy Condition 1 with A^0 and A^j replaced by $\tilde{A}^0$ and $\tilde{A}^j$, respectively. Suppose, furthermore, that the boundary matrix $\tilde{A}(t, x, u; \nu(x))$ is nonnegative on $\boldsymbol{R}^m$ for $(t, x) \in [0, T] \times \partial\Omega$ and $u \in \mathcal{N}(M(x)) \cap \mathcal{O}$. Then (2) is called a linked system of $(1)_1$ if there exists a first order differential operator

$$S(t, x, u; \partial_x) = \sum_{j=1}^{n} S^j(t, x, u)\partial_j$$

satisfying

$$\begin{aligned} S(t, x, u; \xi)A(t, x, u; \xi)A^0(t, x, u)^{-1} \\ = \tilde{A}(t, x, u; \xi)\tilde{A}^0(t, x, u)^{-1}S(t, x, u; \xi) \end{aligned} \tag{3}$$

for $(t, x, u) \in [0, T] \times \bar{\Omega} \times \mathcal{O}$ and $\xi \in \boldsymbol{R}^n$. $S(t, x, u; \partial_x)$ is called a linkage operator corresponding to the linked system (2).

The relation (3) may be rewritten as

$$S(t, x, u; \xi)A^0(t, x, u) = \tilde{A}^0(t, x, u)\tilde{S}(t, x, u; \xi), \tag{4$_1$}$$

$$S(t, x, u; \xi)A(t, x, u; \xi) = \tilde{A}(t, x, u; \xi)\tilde{S}(t, x, u; \xi), \tag{4$_2$}$$

where $(4)_1$ is understood as the definition of $\tilde{S}(t, x, u; \xi)$. $\tilde{S}(t, x, u; \partial_x)$ is a first order differential operator and is called the modified linkage operator attached to $S(t, x, u; \partial_x)$.

Condition 3. There exist a positive integer N and an N-tuple of liked systems of $(1)_1$ with the following properties: Let $(t, x) \in [0, T] \times \partial\Omega$ and let $u \in \mathcal{N}(M(x)) \cap \mathcal{O}$. Let $\tilde{S}_i\,(t, x, u; \partial_x)$ be the modified linkage operator attached to a suitably chosen linkage operator corresponding to the i-th linked system. Then, if $A(t, x, u; \nu(x))v=0$ and $\tilde{S}_i\,(t, x, u; \nu(x))v=0$, $i=1, \cdots, N$, for $v \in \boldsymbol{R}^m$, we have $v=0$.

Let $H^l(\Omega)$ be the usual Sobolev space (on Ω) of order l, with the norm $\|\cdot\|_l$. We denote by $L^k_\infty([0, T]; H^l(\Omega))$ the space of all functions $u=u(t, x)$ such that $\partial^i_t u$, $0 \le i \le k$, are essentially bounded, strongly measurable functions on $[0, T]$ taking values in $H^l(\Omega)$. We set

$$\begin{gathered} X^l(T, \Omega) = \bigcap_{k=0}^{l} L^k_\infty([0, T]; H^{l-k}(\Omega)), \\ |||u|||_{l,T} = \sup_{0 \le t \le T} |||u(t)|||_l, \qquad |||u(t)|||_l^2 = \sum_{k=0}^{l} \|\partial^k_t u(t)\|^2_{l-k}. \end{gathered} \tag{5}$$

Our main result is then stated as follows.

Theorem 1. *Assume Conditions* 1, 2, *and* 3. *Let* $s \ge [n/2]+2$. *Suppose that* $u_0 \in H^s(\Omega)$, $u_0(x) \in \mathcal{O}$ *for* $x \in \bar{\Omega}$ *and that* u_0 *satisfies the compatibility conditions up to order* $s-1$. *Then there exists a positive constant* T_0 *such that the problem* $(1)_{1\text{-}3}$ *has a unique solution* $u \in X^s(T_0, \Omega)$ *satisfying* $u(t, x) \in \mathcal{O}$ *for* $(t, x) \in [0, T_0] \times \bar{\Omega}$.

Remark. An analogous result can be shown for an unbounded domain Ω with smooth, compact boundary $\partial\Omega$ by suitable modification of the conditions.

3. Linearized problem. We study the linearized problem of $(1)_{1\text{-}3}$:

$$(6)_1 \qquad A^0(t,x,u)v_t+\sum_{j=1}^{n} A^j(t,x,u)v_{x_j}=g(t,x) \qquad \text{in } [0,T]\times\Omega,$$

$$(6)_2 \qquad M(x)v=0 \qquad \text{on } [0,T]\times\partial\Omega,$$

$$(6)_3 \qquad v(0,x)=v_0(x) \qquad \text{for } x\in\Omega.$$

Let $K\subset\mathcal{O}$ be a convex compact set. Let M_{s-1} and M_s be positive constants. We denote by $X^s(T,\Omega;K,M_{s-1},M_s)$ the set of all functions u satisfying the following conditions.

$$(7) \qquad \begin{aligned} &u\in X^s(T,\Omega), \quad M(x)u=0 \qquad \text{on } [0,T]\times\partial\Omega,\\ &u(t,x)\in K \qquad \text{for } (t,x)\in[0,T]\times\bar{\Omega},\\ &|||u|||_{s-1,T}\le M_{s-1}, \qquad |||u|||_{s,T}\le M_s. \end{aligned}$$

Theorem 1 is shown by iteration based on the following existence and regularity result for the linearized problem.

Proposition 2. *Assume Conditions* 1, 2, *and* 3. *Let* $s\ge[n/2]+2$ *and let* $1\le l\le s$. *Let* $u\in X^s(T,\Omega;K,M_{s-1},M_s)$ *and let* $g\in H^l([0,T]\times\Omega)$ *(the usual Sobolev space on* $[0,T]\times\Omega$*). Suppose that* $v_0\in H^l(\Omega)$ *and that* v_0 *satisfies the compatibility conditions up to order* $l-1$. *Then the problem* $(6)_{1\text{-}3}$ *has a unique solution* $v\in X^l(T,\Omega)$ *satisfying*

$$(8) \qquad \begin{aligned} |||v(t)|||_l\le{}& C(M_{s-1})e^{C(M_s)t}|||v(0)|||_l+C(M_{s-1})|||g(t)|||_{l-1}\\ &+C(M_s)\int_0^t e^{C(M_s)(t-\tau)}|||g(\tau)|||_l\,d\tau \end{aligned}$$

for $t\in[0,T]$. *Here* $C(M_k)$, $k=s-1,s$, *denote constants depending on* M_k.

The existence of solution to the linearized problem $(6)_{1\text{-}3}$ is proved by the method of noncharacteristic regularization. (See Rauch [4] and Schochet [5].) Therefore, for the proof of Proposition 2, it suffices to show the following a priori estimate.

Proposition 3. *Assume Conditions* 1, 2 *(with maximal nonnegativity replaced by nonnegativity), and* 3. *Let* $s\ge[n/2]+2$ *and let* $1\le l\le s$. *Suppose that* $u\in X^s(T,\Omega;K,M_{s-1},M_s)$ *and that* $g\in H^l([0,T]\times\Omega)$. *Then a solution* $v\in X^{l+1}(T,\Omega)$ *of the problem* $(6)_{1,2}$ *satisfies the inequality* (8) *for* $t\in[0,T]$.

Proof. We first prove (8) under the additional assumptions that $\Omega=\boldsymbol{R}^n_+=\{x_n>0\}$, M is a constant matrix, and the support of $v(t)$ is contained in $\{|x|\le 1,\ 0\le x_n\le\delta_0\}$ for $t\in[0,T]$. Here δ_0 is a small positive constant depending on M_{s-1} which will be specified later. Let us denote by $\partial^k v$ the tangential derivatives (i.e., the derivatives with respect to t and x_j, $j=1,\cdots,n-1$) of order k. Put $\|[v(t)]\|_l^2=\sum_{k=0}^{l}\|\partial^k v(t)\|^2$, where $\|\cdot\|$ stands for the $L^2(\Omega)$-norm. By virtue of Conditions 1 and 2, we can use the standard energy method to obtain

$$(9) \qquad \begin{aligned} \|[v(t)]\|_l\le{}& Ce^{C(M_s)t}\|[v(0)]\|_l\\ &+C(M_s)\int_0^t e^{C(M_s)(t-\tau)}(\|[g(\tau)]\|_l+|||v(\tau)|||_l)d\tau. \end{aligned}$$

Here and in what follows, C denotes a constant independent of M_{s-1} and M_s. To get estimates for the normal derivatives, we use the linkage operators $S_i(t, x, u; \partial_x)$, $i=1, \cdots, p$. Let $\tilde{S}_i(t, x, u; \partial_x)$ be the modified linkage operators and put

$$(10) \qquad w_i = \tilde{S}_i(t, x, u; \partial_x)v \equiv \sum_{j=1}^{n} \tilde{S}_i^j(t, x, u)v_{x_j}, \qquad i=1, \cdots, N.$$

Multiplying both sides of $(6)_1$ on the left by $S_i(t, x, u; \partial_x)$, we see that each w_i satisfies the linked system whose right member is a function depending on (t, x, u) as well as the first order derivatives $\partial_t u, \partial_x u, \partial_t v, \partial_x v$, and $\partial_x g$. Therefore, by applying the standard energy method, we obtain

$$(11) \qquad |||w_i(t)|||_{l-1} \leq Ce^{C(M_s)t} |||w_i(0)|||_{l-1} + C(M_s) \int_0^t e^{C(M_s)(t-\tau)} (|||g(\tau)|||_l + |||v(\tau)|||_l) d\tau,$$

where $i=1, \cdots, N$. Finally we regard $(6)_1$ and (10), $i=1, \cdots, N$, as a system of linear equations for the normal derivative $\partial_n v$. By virtue of Condition 3, this is solved for $\partial_n v$ on $\{|x| \leq 1, 0 \leq x_n \leq \delta_0\}$ with a small positive constant $\delta_0 = \delta_0(M_{s-1})$, and $\partial_n v$ is expressed in terms of $(t, x, u), g, w_i$, and the first order tangential derivatives ∂v. Using this expression, we obtain the following estimate:

$$(12) \qquad |||v(t)|||_l \leq C(M_{s-1})(|||g(t)|||_{l-1} + \|[v(t)]\|_l + \sum_{i=1}^{p} |||w_i(t)|||_{l-1}).$$

The desired estimate (8) follows from (9), (11), (12), and Gronwall's inequality. The general case can be reduced essentially to the case where the additional assumptions are satisfied. This is carried out by making use of a partition of unity and changes of the dependent and independent variables. Since the arguments are standard ones, we omit the details.

References

[1] A. Majda and S. Osher: Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary. Comm. Pure Appl. Math., **28**, 607–675 (1975).

[2] T. Ohkubo: Regularity of solutions to hyperbolic mixed problems with uniformly characteristic boundary. Hokkaido Math. J., **10**, 93–123 (1981).

[3] ——: Well-posedness for quasi-linear hyperbolic mixed problems with characteristic boundary (to appear).

[4] J. Rauch: Symmetric positive systems with boundary characteristic of constant multiplicity. Trans. Amer. Math. Soc., **291**, 167–187 (1985).

[5] S. Schochet: The compressible Euler equations in a bounded domain. Existence of solutions and the incompressible limit. Commun. Math. Phys., **104**, 49–75 (1986).

[6] M. Tsuji: Regularity of solutions of hyperbolic mixed problems with characteristic boundary. Proc. Japan Acad., **48A**, 719–724 (1972).