64. A Generalization of Lefschetz Theorem

By Takao FUJITA
Department of Mathematics, University of Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., June 9, 1987)

We improve the classical Lefschetz theorem as follows:

Theorem. *Let A be an effective ample divisor on an algebraic variety V defined over $\boldsymbol{C}$ of dimension n, let v be a point on $V-A$ such that $V-A-v$ is smooth and set $U=V-v$. Then the relative homotopy group $\pi_k(U, A)$ vanishes for every $k<n$.*

Using Morse theory, we prove this theorem by modifying Andreotti-Frankel method (cf. [1], [2]). First, replacing A by mA for $m\gg 1$ if necessary, we may assume that A is very ample. Thus $V\subset \boldsymbol{P}^N$ and $A=V\cap S$ for some hyperplane S in $\boldsymbol{P}^N$. We fix an affine linear coordinate of $\boldsymbol{P}^N-S\simeq \boldsymbol{C}^N$ and let δ denote the Euclid distance with respect to this coordinate. Set $N_R=\{x\in V-A \mid \delta(x, v)\leqq R\}$ and $U_R=V-N_R$ for each $R>0$. If $r>0$ is small enough, the function $d(x)=\delta(x, v)$ has no critical point in N_{4r}. Hence U_{3r} and U_r are deformation retracts of U.

For a point p in $\boldsymbol{P}^N-S-V$, let f be the function $\delta(x, p)^2$ on $U-A$. By [2; Theorem 6.6], f has no degenerate critical points for almost all p. In particular, we can choose p such that $\delta(p, v)<r$. Set $T_a=A\cup\{x\in V-A \mid f(x)\geqq a^2\}$. Then $T_L\subset U_{3r}\subset T_{2r}\subset U_r$ for any $L\gg 1$. Using Morse theory similarly as in [2; p. 42], we infer that T_{2r} has the homotopy type of T_L with finitely many cells of real dimension $\geqq n+1$ attached, so we obtain $\pi_k(T_{2r}, A)\simeq \pi_k(T_L, A)\simeq\{1\}$ for $k<n$. On the other hand, the composition $\pi_k(U_{3r}, A)\to\pi_k(T_{2r}, A)\to\pi_k(U_r, A)\simeq\pi_k(U, A)$ is bijective. Hence $\pi_k(U, A)$ is trivial. Thus we complete the proof.

Corollary. *Let L be the total space of an ample line bundle on a compact complex manifold M and let X be a compact analytic subspace of L of pure dimension $n=\dim M$. Then, for the natural map $f: X\to M$,*

1) *$\pi_k(f): \pi_k(X)\to\pi_k(M)$ is bijective if $k<n$ and is surjective if $k=n$.*

2) *$H_k(f): H_k(X; \boldsymbol{Z})\to H_k(M; \boldsymbol{Z})$ is bijective if $k<n$ and is surjective if $k=n$.*

3) *$H^k(f): H^k(M; \boldsymbol{Z})\to H^k(X; \boldsymbol{Z})$ is bijective if $k<n$ and is injective with torsion free cokernel if $k=n$.*

4) *$\mathrm{Pic}(M)\to\mathrm{Pic}(X)$ is bijective if $n>2$ and is injective if $n=2$. When $n=2$, the cokernel is torsion free if $H^1(M, \mathcal{O}_M)\to H^1(X, \mathcal{O}_X)$ is injective.*

Proof. Set $\mathcal{L}=\mathcal{O}_M[L]$, $\mathcal{S}=\mathcal{O}_M\oplus\mathcal{L}$, $P=\boldsymbol{P}(\mathcal{S})$ and $H=\mathcal{O}_P(1)$. Then P is a $\boldsymbol{P}^1$-bundle over M and there are disjoint sections M_∞ and M_0 corresponding to quotient bundles $\mathcal{O}_M$ and $\mathcal{L}$ of $\mathcal{S}$, respectively. The open set $P-M_\infty$ is naturally isomorphic to L and M_0 is identified with the 0-section. So we may assume that X is a divisor in P with $X \cap M_\infty = \emptyset$. This implies $X \in |dH|$ for $d = \deg(f)$. Since L is ample, M_∞ can be contracted to a normal point v on another variety V. Then X is mapped isomorphically onto an ample divisor on V. So, by the Theorem, $\pi_k(X) \to \pi_k(V-v) \simeq \pi_k(P-M_\infty) \simeq \pi_k(L) \simeq \pi_k(M)$ is bijective for $k<n$ and is surjective for $k=n$. Thus we prove 1). The other assertions follow from this by standard arguments.

Remark. Let $f: X \to M$ be a finite cyclic covering of compact complex manifolds with branch locus B. Then the above results apply to f if B is ample. Indeed, it is well known that X can be embedded in the total space of a line bundle L on M such that B is a member of $|dL|$, where $d = \deg(f)$.

Conjecture. *Let V, A be as in the theorem and assume that $V-A-\Sigma$ is smooth for some finite set $\Sigma \subset V-A$. Then $\pi_k(V-\Sigma, A) = \{1\}$ for $k<n$.*

Idea of Proof. Fix a coordinate of $\boldsymbol{P}^N - S \simeq \boldsymbol{C}^N$ as above and let δ denote the distance again. For each $R>0$ and each point v_j of Σ, let $N_{j,R} = \{x \in V-A \mid \delta(x, v_j) \leqq R\}$ and set $U_R = V - \bigcup_j N_{j,R}$. Take a sufficiently small $r>0$ such that U_a is a deformation retract of U for any $a<4r$. For each v_j, take a point p_j off V with $\delta(p_j, v_j) < r$ and set $g(x) = \Sigma_j \delta(x, p_j)^{-2}$ for $x \in V-A$ and $g(x)=0$ for $x \in A$. Perhaps g has no degenerate critical point on $U_r - A$ for suitably chosen p_j's (this part requires a proof). Set $T = \{x \in V \mid g(x) < 1/4r^2\}$. Then $U_{3r} \subset T \subset U_r$ since r is sufficiently small. Since $\partial^2 g / \partial_\alpha \bar{\partial}_\beta = 0$ at any critical point of g, the Hessian matrix with respect to some real parameter is of the form $\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}$, where X and Y are symmetric matrices. In particular its signature is (n, n). So we have $\pi_k(T, A) = \{1\}$ by Morse theory similarly as in the classical case. This implies $\pi_k(U, A) = \{1\}$.

References

[1] A. Andreotti and T. Frankel: The Lefschetz theorem on hyperplane sections. Ann. of Math., **69**, 713–717 (1959).

[2] J. Milnor: Morse theory. Ann. of Math. Studies, **51**, Princeton Univ. Press (1963).