63. Casson's Invariant for Homology 3-Spheres and Characteristic Classes of Surface Bundles

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(Communicated by Kunihiko KODAIRA, M. J. A., June 9, 1987)

1. Introduction. Let $V=S^1\times D^2$ be a (framed) solid torus and choose two disjoint embedded discs D_-, D_+ in ∂V. Define an oriented 3-dimensional handlebody H_g of genus g by $H_g=V_1\natural\cdots\natural V_g$ (g-copies of V) where D_- of V_i is attached to D_+ of V_{i-1}. We denote Σ_g for ∂H_g which has an embedded disc $D^2=D_+$ of V_g. Let $\mathcal{I}_{g,1}$ be the Torelli group of Σ_g rel. D^2. Now let ι_g be a diffeomorphism of Σ_g defined as $\iota_g=\prod_{i=1}^g \rho_i$ where $\rho_i=\varphi_l\varphi_m\varphi_l$, φ_l and φ_m being, respectively, the Dehn twist on the longitude and meridian curves of the framed torus ∂V_i, so that $H_g\cup_{\iota_g}(-H_g)=S^3$. For each element $\varphi\in\mathcal{I}_{g,1}$, the manifold $M(\varphi)=H_g\cup_{\iota_g\varphi}(-H_g)$ is an oriented homology 3-sphere and we have the Casson invariant $\lambda(M(\varphi))\in \boldsymbol{Z}$. Thus we have a map $\lambda:\mathcal{I}_{g,1}\to\boldsymbol{Z}$. The purpose of the present note is to announce our result concerning the map λ. Briefly speaking we have shown that the Casson invariant is a kind of secondary invariant associated with the characteristic classes of surface bundles introduced in [7]. As a result we have obtained an alternative definition of λ (see Theorems 6 and 7).

2. Johnson's homomorphisms. Let $x_1,\cdots,x_g,y_1,\cdots,y_g$ be the symplectic basis of $H=H_1(\Sigma_g;\boldsymbol{Z})$ such that x_i and y_i are represented by the longitude and meridian of ∂V_i, respectively. Consider the basis $x_i\wedge y_j$ $(i,j=1,\cdots,g)$, $x_i\wedge x_j$ $(i<j)$, $y_i\wedge y_j$ $(i<j)$ of $\bigwedge^2 H$ and write t_i $(i=1,\cdots,\binom{2g}{2})$ for these elements (in any order). Let T be the submodule of $\bigwedge^2 H\otimes\bigwedge^2 H\subset\bigwedge^2 H\otimes H^2$ generated by $t_i\otimes t_i$ and $t_i\otimes t_j+t_j\otimes t_i$ $(i\neq j)$. Hereafter we simply write $t_i\leftrightarrow t_j$ for $t_i\otimes t_j+t_j\otimes t_i$. Let $\bar{T}$ be the image in $(\bigwedge^2 H\otimes H/\bigwedge^3 H)\otimes H$ of T under the projection $\bigwedge^2 H\otimes H^2\to(\bigwedge^2 H\otimes H/\bigwedge^3 H)\otimes H$. Then we have Johnson's homomorphisms:

$$\tau_2:\mathcal{I}_{g,1}\longrightarrow\textstyle\bigwedge^3 H$$
$$\tau_3:\mathcal{K}_{g,1}\longrightarrow\bar{T}$$

where $\mathcal{K}_{g,1}$ is the subgroup of $\mathcal{I}_{g,1}$ generated by Dehn twists on bounding simple closed curves (see [4], [5], [6], [9] for details). Define a homomorphism $\theta_0:T\to\boldsymbol{Z}$ by requiring the value of it on each element of the basis of T described above as $\theta_0(x_i\wedge x_j\leftrightarrow y_i\wedge y_j)=1$ and θ_0 (other element)$=0$.

3. Characteristic classes of surface bundles. Here we begin by briefly recalling several results from our previous papers [7], [8], [9]. Let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g relative to D^2 and let $e_1\in H^2(\mathcal{M}_{g,1};\boldsymbol{Z})$ be the first characteristic class of surface bundles. We constructed a crossed homomorphism $k:\mathcal{M}_{g,1}\to H$, which is uniquely defined up to coboundaries, and proved that the cohomology class e_1 is represented by the 2-cocycle c of $\mathcal{M}_{g,1}$ given by $c(\varphi, \psi)=k(\varphi)\cdot k(\psi^{-1})$ $(\varphi, \psi \in \mathcal{M}_{g,1})$. We also proved that $i^*(e_1)=0$ in $H^2(\mathcal{I}_{g,1})$ where $i: \mathcal{I}_{g,1} \to \mathcal{M}_{g,1}$ is the inclusion. (Formerly we have only proved that $i^*(e_1)$ is a torsion class, but Johnson's result [6] and the form of the cocycle c above imply that $i^*(e_1)$ is actually zero.) Hence there exists a map

$$d: \mathcal{I}_{g,1} \longrightarrow \boldsymbol{Z}$$

such that $i^*(c)=\delta d$. It is well defined up to elements of $H^1(\mathcal{I}_{g,1}; \boldsymbol{Z})$. Also $\operatorname{Im} d$ is contained in $2\boldsymbol{Z}$.

Theorem 1. *Let $\varphi \in \mathcal{K}_{g,1}$ be a Dehn twist on a bounding simple closed curve of genus k in $\Sigma_g \backslash D^2$. Then we have $d(\varphi)=-4k(k-1)$.*

4. Statement of the main results. Let $\mathcal{L}_{g,1}=\operatorname{Ker} \tau_3$. It is the subgroup of $\mathcal{M}_{g,1}$ consisting of all elements which act on $\pi_1(\Sigma_g \backslash \mathring{D}^2)$ trivially modulo four-fold commutators.

Theorem 2. $\lambda=(1/24)d$ *on* $\mathcal{L}_{g,1}$.

As a corollary to this theorem, we can answer a problem of Johnson ([5], p. 172, Problem B) negatively.

Corollary 3. *The three-fold commutator subgroup of $\mathcal{I}_{g,1}$, which is a normal subgroup of $\mathcal{L}_{g,1}$ has an infinite index in $\mathcal{L}_{g,1}$.*

Now let $\mathcal{N}_{g,1}$ be the subgroup of $\mathcal{M}_{g,1}$ consisting of isotopy classes of diffeomorphisms of Σ_g which can be extended to those of H_g. Define an equivalence relation $\sim$ on $\mathcal{I}_{g,1}$ as follows. Two elements φ and $\psi \in \mathcal{I}_{g,1}$ are equivalent iff there are elements $\xi_1, \xi_2 \in \mathcal{N}_{g,1}$ such that $\psi=\iota_g^{-1}\xi_1\iota_g\varphi\xi_2$. Then the classical Heegaard-Reidemeister-Singer theorem implies that

$$\lim_{g\to\infty} \mathcal{I}_{g,1}/\sim\,=\mathcal{H}(3),$$

where $\mathcal{H}(3)$ denotes the set of all diffeomorphism classes of oriented homology 3-spheres. The correspondence is given by $\mathcal{I}_{g,1} \ni \varphi \mapsto M(\varphi) \in \mathcal{H}(3)$. Now let W_x (resp. W_y) be the submodule of $\wedge^3 H$ generated by the elements $x_i \wedge x_j \wedge y_k$ and $x_i \wedge x_j \wedge x_k$ (resp. $x_i \wedge y_j \wedge y_k$ and $y_i \wedge y_j \wedge y_k$) so that $\wedge^3 H= W_x \oplus W_y$. It is easy to deduce from the result of Suzuki [10] that for any element $u \in W_y$, there exists an element $\varphi \in \mathcal{I}_{g,1} \cap \mathcal{N}_{g,1}$, such that $\tau_2(\varphi)=u$. Using this fact we can prove

Proposition 4. *For any element $\varphi \in \mathcal{I}_{g,1}$, there exist elements $\psi_1, \psi_2 \in \mathcal{I}_{g,1} \cap \mathcal{N}_{g,1}$ such that $\iota_g^{-1}\psi_1\iota_g\varphi\psi_2$ is contained in $\mathcal{K}_{g,1}$.*

Corollary 5. $\mathcal{H}(3)=\lim_{g\to\infty} \mathcal{K}_{g,1}/\sim$.

This simple result might be useful when one tries to obtain new invariants for homology 3-spheres from those of knots in S^3.

Next we define a homomorphism $\bar{d}: T \to \boldsymbol{Z}$ by requiring $\bar{d}(t_i \otimes t_i)=0$ and $\bar{d}(a\wedge b \leftrightarrow c \wedge d)=(a\cdot b)(c\cdot d)-(a\cdot c)(b\cdot d)+(a\cdot d)(b\cdot c)$ $(a, b, c, d \in H)$. For each element $\varphi \in \mathcal{I}_{g,1}$, choose $\psi_1, \psi_2 \in \mathcal{I}_{g,1} \cap \mathcal{N}_{g,1}$ such that $\tilde{\varphi}=\iota_g^{-1}\psi_1\iota_g\varphi\psi_2$ is contained in $\mathcal{K}_{g,1}$ (see Proposition 4) and also choose $t \in T$ such that $\tau_3(\tilde{\varphi})=\bar{t} \in \bar{T}$, where $\bar{t}$ is the projection of t. Now set

$$\delta(\varphi)=\theta_0(t)+\frac{1}{3}\left\{\frac{1}{8}d(\mathring{\varphi})+\bar{d}(t)\right\}.$$

Theorem 6. *The value $\delta(\varphi)$ depends only on the element φ, namely it is independent of the choices of ψ_1, ψ_2 and t. Moreover if $\varphi \sim \psi$, then $\delta(\varphi) = \delta(\psi)$ so that we have a map $\delta: \mathcal{H}(3)\to \boldsymbol{Z}$.*

For the proof of the above theorem, the determination of a finite set of normal generators for the group $\mathcal{J}_{g,1}\cap \mathfrak{N}_{g,1}$, which can be derived from the classical presentation of $GL(g, \boldsymbol{Z})$ and the result of Suzuki [10] along the lines of Birman's paper [1], plays a crucial role. Now we have

Theorem 7. *δ is equal to λ as a map $\mathcal{H}(3)\to \boldsymbol{Z}$.*

Thus Theorem 6 provides a method of computing $\lambda(\varphi)$ and also we have an alternative definition of Casson's invariant in terms of the pasting maps of Heegaard decompositions (of course the point here is that we do not use the existence of λ in the proof of Theorem 6).

5. The Magnus representation of the Torelli group. Choose a free generators $\gamma_1, \cdots, \gamma_{2g}$ of $\pi_1(\Sigma_g\backslash \mathring{D}^2)$ which is a free group of rank $2g$. For each element $\varphi \in \mathcal{J}_{g,1}$, consider the matrix

$$r(\varphi)=\left(\frac{\partial\varphi(\gamma_j)}{\partial\gamma_i}\right) \in GL_{2g}(\boldsymbol{Z}[H])$$

where $(\partial/\partial\gamma_i)$ is the Fox differential with respect to γ_i. This defines a map $r: \mathcal{J}_{g,1}\to GL_{2g}(\boldsymbol{Z}[H])$ which is actually a homomorphism and following Birman's book [2] we call it the Magnus representation of the Torelli group. Let I be the augmentation ideal of $\boldsymbol{Z}[H]$ and let $\boldsymbol{Z}[H]_n$ be the quotient $\boldsymbol{Z}[H]/I^n$. We write r_n for the composition $\mathcal{J}_{g,1}\to GL_{2g}(\boldsymbol{Z}[H])\to GL_{2g}(\boldsymbol{Z}[H]_n)$. The homomorphisms r_n is closely related with Johnson's homomorphisms τ_n. For example

Theorem 8. (i) *$\operatorname{Im} r_2$ is naturally isomorphic to $\wedge^3 H$ so that r_2 is essentially equivalent to τ_2. In particular $\det r(\varphi)=k(\varphi)$ for all $\varphi \in \mathcal{J}_{g,1}$.*

(ii) *$r_3(\mathcal{K}_{g,1})$ is naturally isomorphic to $\operatorname{Im}\tau_3$ so that r_3 restricted to $\mathcal{K}_{g,1}$ is equivalent to τ_3.*

(iii) *$\operatorname{Im} r_3$ is isomorphic to the central extension of $\wedge^3 H$ by $\operatorname{Im}\tau_3: 0\to \operatorname{Im}\tau_3\to \operatorname{Im} r_3\to \wedge^3 H\to 1$, whose Euler class $\in H^2(\wedge^3 H; \operatorname{Im}\tau_3)$ is given by*

$$\begin{aligned}
&\wedge^3 H\times \wedge^3 H \ni (a\wedge b\wedge c,\ d\wedge e\wedge f)\\
&\quad\mapsto (a\cdot d)b\wedge c\leftrightarrow e\wedge f+(a\cdot e)b\wedge c\leftrightarrow f\wedge d+(a\cdot f)b\wedge c\leftrightarrow d\wedge e\\
&\quad +(b\cdot d)c\wedge a\leftrightarrow e\wedge f+(b\cdot e)c\wedge a\leftrightarrow f\wedge d+(b\cdot f)c\wedge a\leftrightarrow d\wedge e \ \in \operatorname{Im}\tau_3\subset \bar{T}.\\
&\quad +(c\cdot d)a\wedge b\leftrightarrow e\wedge f+(c\cdot e)a\wedge b\leftrightarrow f\wedge d+(c\cdot f)a\wedge b\leftrightarrow d\wedge e
\end{aligned}$$

Theorem 6 and Theorem 8 (iii) imply

Theorem 9. *For two elements $\varphi, \psi \in \mathcal{J}_{g,1}$, we have*

$$\lambda(\varphi\psi)=\lambda(\varphi)+\lambda(\psi)-2m$$

where m is defined as follows. Write $\tau_2(\varphi)=\sum a_{ijk}y_i\wedge y_j\wedge y_k+$other terms, $\tau_2(\psi)=\sum b_{ijk}x_i\wedge x_j\wedge x_k+$other terms, with respect to the basis of $\wedge^3 H$ described before, then $m=\sum a_{ijk}b_{ijk}$.

In particular λ is a homomorphism for $g=2$ and also the above result is consistent with the result of Birman-Craggs [3] that λ mod 2, which is the Rohlin invariant, defines a homomorphism $\mathcal{J}_{g,1} \to \boldsymbol{Z}/2$.

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