62. *Differentiable Vectors and Analytic Vectors in Completions of Certain Representation Spaces of a Kac-Moody Algebra*

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Let $\mathfrak{g}_k$ be a Kac-Moody algebra over a field k with a symmetrizable generalized Cartan matrix, and $\mathfrak{h}_k$ the Cartan subalgebra of $\mathfrak{g}_k$. Here, we concentrate on the case $k=\boldsymbol{R}$ or $\boldsymbol{C}$, the real or complex number field. We denote $\mathfrak{g}_{\boldsymbol{C}}$ and $\mathfrak{h}_{\boldsymbol{C}}$ simply by $\mathfrak{g}$ and $\mathfrak{h}$ respectively, then $\mathfrak{g}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{g}_{\boldsymbol{R}}$, and $\mathfrak{h}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{h}_{\boldsymbol{R}}$. Let $\mathfrak{k}$ be the unitary form of $\mathfrak{g}$ (cf. [2]). Let Λ be a dominant integral element in $\mathfrak{h}_{\boldsymbol{R}}^*$, and $L(\Lambda)$ the irreducible highest weight module for $\mathfrak{g}$ with highest weight Λ. We denote by $H(\mathrm{ad})$ and $H(\Lambda)$ the completions of $\mathfrak{g}$ and $L(\Lambda)$ with respect to the standard inner products $(\cdot|\cdot)_1$ and $(\cdot|\cdot)_\Lambda$, respectively. In [2], we defined a group K^Λ associated with $\mathfrak{k}$ as a subgroup of the unitary group on the Hilbert space $H(\Lambda)$ generated by the naturally defined exponentials of elements in $\mathfrak{k}$, and then proved that any element in L(Λ) is differentiable and analytic for actions of the exponentials. In this paper, we extend these results so that we can treat the case of adjoint representation as well. Further we show some properties of the exponentials needed to study fine structures of K^Λ.

§1. Basic facts for Kac-Moody algebras. The notations used here are the same as in [2]. The standard contravariant Hermitian form $(\cdot|\cdot)_0$ on $\mathfrak{g}$ is, unfortunately, not positive definite on $\mathfrak{h}$ in general, though it is always positive definite on each root space $\mathfrak{g}^\alpha$. So, we introduce a new inner product $(\cdot|\cdot)_1$ on $\mathfrak{g}$ as follows: first on $\mathfrak{h}$

$$(h\,|\,h')_0\leqq\|h\|_1\|h'\|_1,\quad \lambda(h)\leqq\|\lambda\|_1\|h\|_1 \text{ for } h,\ h'\in\mathfrak{h},\quad \lambda\in\mathfrak{h}^*,$$

where $\|h\|_1=(h\,|\,h)_1^{1/2}$, and $\|\cdot\|_1$ on $\mathfrak{h}^*$ is the dual of $\|\cdot\|_1$ on $\mathfrak{h}$. Then extend it to the whole space $\mathfrak{g}$ by

$$(x\,|\,x')_1=(x_-\,|\,x'_-)_0+(x_0\,|\,x'_0)_1+(x_+\,|\,x'_+)_0$$

for $x=x_-+x_0+x_+$, $x'=x'_-+x'_0+x'_+\in\mathfrak{g}$ with $x_\pm,\ x'_\pm\in\mathfrak{n}_\pm=\sum_{\alpha\in\Delta_+}\mathfrak{g}^{\pm\alpha}$, and x_0, $x'_0\in\mathfrak{h}$.

Denote by $\underline{\mathfrak{g}}$ the infinite direct product of $\mathfrak{g}^0=\mathfrak{h}$ and of all the root spaces $\mathfrak{g}^\alpha$ $(\alpha\in\Delta)$, and $\underline{L}(\Lambda)$ that of all the weight spaces $L(\Lambda)_\mu$ of $L(\Lambda)$. Each element in $\mathfrak{g}$ acts both on $\underline{\mathfrak{g}}$ and on $\underline{L}(\Lambda)$ naturally. The completions $H(\mathrm{ad})$ and $H(\Lambda)$ of $\mathfrak{g}$ and $L(\Lambda)$ are defined as Hilbert spaces contained in $\underline{\mathfrak{g}}$ and $\underline{L}(\Lambda)$ respectively as:

$$H(\mathrm{ad})=\{(x_\alpha)_\alpha\in\underline{\mathfrak{g}}\,;\ \sum_\alpha\|x_\alpha\|_1^2<+\infty\},\qquad H(\Lambda)=\{(v_\mu)_\mu\in\underline{L}(\Lambda)\,;\ \sum_\mu\|v_\mu\|_\Lambda^2<+\infty\}.$$

§2. Estimates of norms of $\mathfrak{g}$-action. An element h_0 in $\mathfrak{h}_{\boldsymbol{R}}$ is called *strictly dominant* if $\alpha(h_0)>0$ for any positive root α. Fix such an element h_0. Modifying slightly the method in [1, Proposition 3.1], we obtain the following estimates of $\mathfrak{g}$-action on the $\mathfrak{g}$-modules $(\mathfrak{g}, \mathrm{ad})$ and $L(\Lambda)$.

Proposition 2.1. i) *There exists a positive number C_1 such that*

$$\|[x, y]\|_1 \leqq C_1(\|x\|_1 \|[h_0, y]\|_1 + \|[h_0, x]\|_1 \|y\|_1) \qquad (x, y \in \mathfrak{g}).$$

ii) *There exists a positive number $C_{1,\Lambda}$ such that*

$$\|xv\|_\Lambda \leqq C_{1,\Lambda}(\|x\|_1 \|v\|_\Lambda + \|x\|_1 \|h_0 v\|_\Lambda + \|[h_0, x]\|_1 \|v\|_\Lambda) \qquad (x \in \mathfrak{g},\ v \in L(\Lambda)).$$

By this proposition, we get the key estimates as follows.

Proposition 2.2. *For any $x_1, x_2, \cdots, x_m \in \mathfrak{g}$ and $v \in L(\Lambda)$, there hold the inequalities respectively for $(\mathfrak{g}, \mathrm{ad})$ and $L(\Lambda)$:*

i)
$$\|[x_1, [x_2, \cdots, [x_{m-1}, x_m]\cdots]]\|_1 \leqq (m-1)!\, C_1^{m-1} \times \sum_{\substack{p_1, \cdots, p_m \geqq 0 \\ p_1 + \cdots + p_m = m-1}} \prod_j \frac{1}{p_j!} \|(\mathrm{ad}\, h_0)^{p_j} x_j\|_1,$$

ii)
$$\|x_1 x_2 \cdots x_m v\|_\Lambda \leqq (m+1)!\, C_{1,\Lambda}^m \times \sum_{\substack{p_1, \cdots, p_m, q \geqq 0 \\ p_1 + \cdots + p_m + q \leqq m}} \left\{\prod_j \frac{1}{p_j!} \|(\mathrm{ad}\, h_0)^{p_j} x_j\|_1\right\} \frac{1}{q!} \|h_0^q v\|_\Lambda.$$

§ 3. Differentiable vectors and analytic vectors. We define the spaces $H_m(\mathrm{ad})$ (resp. $H_m(\Lambda)$) of vectors of class C^m ($m=0, 1, 2, \cdots, \infty$) in $H(\mathrm{ad})$ (resp. $H(\Lambda)$) inductively as follows:

$$H_0(\mathrm{ad}) = H(\mathrm{ad}), \quad H_0(\Lambda) = H(\Lambda); \quad H_\infty(\mathrm{ad}) = \bigcap_{m \geqq 0} H_m(\mathrm{ad}), \quad H_\infty(\Lambda) = \bigcap_{m \geqq 0} H_m(\Lambda);$$

$$H_m(\mathrm{ad}) = \{y \in H_{m-1}(\mathrm{ad})\,;\ [x, y] \in H_{m-1}(\mathrm{ad}) \text{ for any } x \in \mathfrak{g}\},$$

$$H_m(\Lambda) = \{v \in H_{m-1}(\Lambda)\,;\ xv \in H_{m-1}(\Lambda) \text{ for any } x \in \mathfrak{g}\},$$

and define the spaces $H_\omega(\mathrm{ad})$ and $H_\omega(\Lambda)$ of "analytic" vectors by

$$H_\omega(\mathrm{ad}) = \{y \in H_\infty(\mathrm{ad})\,;\ {}^\forall x \in \mathfrak{g},\ {}^\exists \varepsilon > 0 \text{ such that } \sum_{m \geqq 0} (1/m!)\varepsilon^m \|(\mathrm{ad}\, x)^m y\|_1 < +\infty\},$$

$$H_\omega(\Lambda) = \{v \in H_\infty(\Lambda)\,;\ {}^\forall x \in \mathfrak{g},\ {}^\exists \varepsilon > 0 \text{ such that } \sum_{m \geqq 0} (1/m!)\varepsilon^m \|x^m v\|_\Lambda < +\infty\}.$$

Making use of Proposition 2.2, we can prove the following remarkable fact.

Theorem 3.2. *Let $h_0 \in \mathfrak{h}_{\boldsymbol{R}}$ be any strictly dominant element. Then, the spaces $H_m(\mathrm{ad})$ and $H_m(\Lambda)$ ($m \in \mathcal{M} := \{0, 1, 2, \cdots, \infty, \omega\}$) are characterized by one element h_0 as follows:*

(i1) $H_m(\mathrm{ad}) = \{y \in \mathfrak{g}\,;\ (\mathrm{ad}\, h_0)^m y \in H(\mathrm{ad})\}$ for $0 \leqq m < +\infty$,

(i2) $H_\omega(\mathrm{ad}) = \{y \in H_\infty(\mathrm{ad})\,;\ {}^\exists \varepsilon > 0$ such that $\sum_{m \geqq 0} (1/m!)\varepsilon^m \|(\mathrm{ad}\, h_0)^m y\|_1 < +\infty\}$;

(ii1) $H_m(\Lambda) = \{v \in \underline{L}(\Lambda)\,;\ h_0^m v \in H(\Lambda)$ for $0 \leqq m < +\infty\}$,

(ii2) $H_\omega(\Lambda) = \{v \in H_\infty(\Lambda)\,;\ {}^\exists \varepsilon > 0$ such that $\sum_{m \geqq 0} (1/m!)\varepsilon^m \|h_0^m v\|_\Lambda < +\infty\}$.

§ 4. Topologies on the spaces. $H_m(\mathrm{ad})$ and $H_m(\Lambda)$ ($m \in \mathcal{M}$). Theorem 3.2 enables us to define inner products on the spaces $H_m(\mathrm{ad})$ and $H_m(\Lambda)$ ($0 \leqq m < +\infty$) respectively by

$$(x\,|\,y)_{\mathrm{ad}, m} = \textstyle\sum_{l=0}^m ((\mathrm{ad}\, h_0)^l x\,|\,(\mathrm{ad}\, h_0)^l y)_1 \qquad \text{for } x, y \in H_m(\mathrm{ad}),$$

$$(u\,|\,v)_{\Lambda, m} = \textstyle\sum_{l=0}^m (h_0^l u\,|\,h_0^l v)_\Lambda \qquad \text{for } u, v \in H_m(\Lambda),$$

with which $H_m(\mathrm{ad})$ and $H_m(\Lambda)$ are both Hilbert spaces. On the spaces

$H_\infty(\mathrm{ad})=\bigcap_{m\geqq 0} H_m(\mathrm{ad})$ and $H_\infty(\Lambda)=\bigcap_{m\geqq 0} H_m(\Lambda)$, we introduce projective limit topologies of the topologies of these Hilbert spaces.

By Proposition 2.2, the bracket product $\mathfrak{g}\times\mathfrak{g}\ni(x, y)\mapsto[x, y]\in\mathfrak{g}$ and the map $\mathfrak{g}\times L(\Lambda)\ni(x, v)\mapsto xv\in L(\Lambda)$ can be extended to continuous bilinear maps from $H_m(\mathrm{ad})\times H_m(\mathrm{ad})$ into $H_{m-1}(\mathrm{ad})$ and $H_m(\mathrm{ad})\times H_m(\Lambda)$ into $H_{m-1}(\Lambda)$ respectively, for any $m>0$. In particular, we have a topological Lie algebra $H_\infty(\mathrm{ad})$, denoted also by $\mathfrak{g}_\infty$, and its continuous representation on $H_\infty(\Lambda)$.

Now, we consider topologies on the spaces $H_\omega(\mathrm{ad})$ and $H_\omega(\Lambda)$. For $0<\varepsilon\leqq+\infty$, define subspaces of $H_\omega(\mathrm{ad})$ and of $H_\omega(\Lambda)$ respectively by

$$H_\omega(\mathrm{ad}\,;\varepsilon)=\{y=H_\omega(\mathrm{ad})\,;\,0<{}^\forall\delta<\varepsilon,\ \|y\|_{\mathrm{ad},\omega,\delta}=\sum_{m\geqq 0}(1/m\,!)\delta^m\,\|(\mathrm{ad}\,h_0)^m y\|_1<+\infty\},$$

$$H_\omega(\Lambda\,;\varepsilon)=\{v\in H_\omega(\Lambda)\,;\,0<{}^\forall\delta<\varepsilon,\ \|v\|_{\Lambda,\omega,\delta}=\sum_{m\geqq 0}(1/m\,!)\delta^m\,\|h_0^m v\|_\Lambda<+\infty\}.$$

The space $H_\omega(\mathrm{ad}\,;\varepsilon)$ with a family of norms $\|\cdot\|_{\mathrm{ad},\omega,\delta}$ $(0<\delta<\varepsilon)$ is a Fréchet space. Similarly, so is $\{H_\omega(\Lambda\,;\varepsilon),\ \|\cdot\|_{\Lambda,\omega,\delta}\ (0<\delta<\varepsilon)\}$. We see that $H_\omega(\mathrm{ad})=\bigcup_{0<\varepsilon\leqq+\infty}H_\omega(\mathrm{ad}\,;\varepsilon)$ and $H_\omega(\Lambda)=\bigcup_{0<\varepsilon\leqq+\infty}H_\omega(\Lambda\,;\varepsilon)$, and so we adopt inductive limit topologies on the spaces $H_\omega(\mathrm{ad})$ and $H_\omega(\Lambda)$. Thanks to Proposition 2.1, we see that for any $0<\varepsilon\leqq+\infty$, $H_\omega(\mathrm{ad}\,;\varepsilon)$ is a Fréchet Lie algebra, denoted also by $\mathfrak{g}_{\omega,\varepsilon}$, and that $\mathfrak{g}_{\omega,\varepsilon}$ leaves $H_\omega(\Lambda\,;\varepsilon)\subset H_\infty(\Lambda)$ invariant and acts continuously on it. Consequently, $H_\omega(\mathrm{ad})$ is a topological Lie algebra, denoted also by $\mathfrak{g}_\omega$, and acts continuously on $H_\omega(\Lambda)$.

§5. Completions of the unitary form and their exponentials. Since the $*$-operation on $\mathfrak{g}$ preserves the norm $\|\cdot\|_1$ and leaves invariant the element h_0, it extends to each of the spaces $H_m(\mathrm{ad})$ $(m\in\mathcal{M})$ and $H_\omega(\mathrm{ad}\,;\varepsilon)$ $(0<\varepsilon\leqq+\infty)$ by continuity. So, we can define the completions of the unitary form $\mathfrak{k}\subset\mathfrak{g}$ in respective spaces by

$$H_m^u(\mathrm{ad})=\{y\in H_m(\mathrm{ad})\,;\,y+y^*=0\},\quad H_\omega^u(\mathrm{ad}\,;\varepsilon)=\{y\in H_\omega(\mathrm{ad}\,;\varepsilon)\,;\,y+y^*=0\}.$$

In particular, we get real Lie subalgebras $\mathfrak{k}_\infty=H_\infty^u(\mathrm{ad})$ (resp. $\mathfrak{k}_\omega=H_\omega^u(\mathrm{ad})$ and $\mathfrak{k}_{\omega,\varepsilon}=H_\omega^u(\mathrm{ad}\,;\varepsilon)$) of $\mathfrak{g}_\infty$ (resp. $\mathfrak{g}_\omega$ and $\mathfrak{g}_{\omega,\varepsilon}$) which are topologically closed.

Utilizing the criterion in [4, Chap. IX] for exponentiability of closed operators on locally convex topological vector spaces, we get

Theorem 5.1. *Let* $x=\sum_{\alpha\in\Delta\cup\{0\}}x_\alpha\in H_1^u(\mathrm{ad})$.

(i) *There exists a unique* 1*-parameter group* $\exp t(\mathrm{ad}\,x)=e^{t(\mathrm{ad}\,x)}$ *of bounded operators on the Hilbert space* $H(\mathrm{ad})$ *such that*

$$(d/dt)\{(\exp t(\mathrm{ad}\,x))y\}=(\exp t(\mathrm{ad}\,x))[x, y]\qquad \textit{for any } y\in H_1(\mathrm{ad}).$$

Moreover, the operator norm $\|\cdot\|_{op}$ *of* $\exp t(\mathrm{ad}\,x)$ *is evaluated as*

$$\|\exp t(\mathrm{ad}\,x)\|_{op}\leqq\exp(2\,|t|\,(\textstyle\sum_{\alpha\in\Delta}\|\alpha\|_1^2\,\|x_\alpha\|_1^2)^{1/2})\qquad \textit{for all } t\in\boldsymbol{R}.$$

(ii) *There exists a unique* 1*-parameter group* $\exp tx=e^{tx}$ *of unitary operators on the Hilbert space* $H(\Lambda)$ *such that*

$$(d/dt)\{(\exp tx)v\}=(\exp tx)xv\qquad \textit{for any } v\in H_1(\Lambda).$$

Let $m\in\mathcal{M}$ and $x\in H_m^u(\mathrm{ad})$. In the case $0<m<+\infty$, each element in $H_m(\mathrm{ad})$ or in $H_m(\Lambda)$ is exactly m-times differentiable for the 1-parameter group $\exp t(\mathrm{ad}\,x)$ or $\exp tx$ in usual sense. As for the case $m=\omega$, each element in $H_\omega(\mathrm{ad})$ or in $H_\omega(\Lambda)$ is really analytic as well.

§6. Properties of the exponential map on $\mathfrak{k}_\omega$. Each element in $\mathfrak{g}$ is analytic and acts on $\mathfrak{g}$ and $\underline{L}(\Lambda)$, and so we get

Lemma 6.1. *Let* $0<\delta<\varepsilon\leqq+\infty$, $x\in\mathfrak{k}_{\omega,\varepsilon}$.

i) *If* $C_1\|x\|_{\mathrm{ad},\omega,\delta}<\delta$, *then there holds the equality*

$$[y, e^{(\mathrm{ad}\,x)}z]=e^{(\mathrm{ad}\,x)}[e^{-(\mathrm{ad}\,x)}y, z] \qquad \text{for any } z\in H_1(\mathrm{ad}),\ y\in\mathfrak{g}.$$

ii) *If* $\max(C_1, C_{1,\Lambda})\|x\|_{\mathrm{ad},\omega,\delta}<(1+\delta^{-1})^{-1}$, *then*

$$ye^x v=e^x(e^{-(\mathrm{ad}\,x)}y)v \qquad \text{for any } v\in H_1(\Lambda),\ y\in\mathfrak{g}.$$

Taking h_0 as y in this lemma and using Proposition 2.2, we obtain

Theorem 6.2. *Let* $x\in\mathfrak{k}_\omega$ *and* $m\in\mathcal{M}$. *The operator* $e^{(\mathrm{ad}\,x)}$ *(resp.* e^x*) leaves invariant each* $H_m(\mathrm{ad})$ *(resp.* $H_m(\Lambda)$*), and the restriction of* $e^{(\mathrm{ad}\,x)}$ *(resp.* e^x*) to* $H_m(\mathrm{ad})$ *(resp.* $H_m(\Lambda)$*) is continuous.*

By this theorem, Lemma 6.1 is improved as follows.

Proposition 6.3. *Let* $x\in\mathfrak{k}_\omega$.

i) $[y, e^{(\mathrm{ad}\,x)}z]=e^{(\mathrm{ad}\,x)}[e^{-(\mathrm{ad}\,x)}y, z]$ *for any* $y\in H_1(\mathrm{ad})$, $z\in H_1(\mathrm{ad})$.

ii) $ye^x v=e^x(e^{-(\mathrm{ad}\,x)}y)v$ *for any* $y\in H_1(\mathrm{ad})$, $v\in H_1(\Lambda)$.

From this proposition, we obtain

Proposition 6.4. *For any* $x\in\mathfrak{k}_\omega$ *and* $y\in H_1^u(\mathrm{ad})$, *it holds that*

i) $$e^{(\mathrm{ad}\,x)}e^{(\mathrm{ad}\,y)}e^{-(\mathrm{ad}\,x)}=e^{\mathrm{ad}\,(e^{(\mathrm{ad}\,x)}y)},$$

ii) $$e^x e^y e^{-x}=e^{(e^{(\mathrm{ad}\,x)}y)}.$$

Let K_ω^{ad} (resp. K_ω^Λ) be the group of operators on $H(\mathrm{ad})$ (resp. $H(\Lambda)$) generated by $\exp(\mathrm{ad}\,\mathfrak{k}_\omega)$ (resp. $\exp\mathfrak{k}_\omega$). Since the map exp from $\mathfrak{k}_\omega$ into the unitary group on $H(\Lambda)$ equipped with the strong operator topology, is continuous, as we can prove by using Theorem 6.2 and the differentiability of elements in $H_1(\Lambda)$, K_ω^Λ is a natural subgroup of the group K^Λ defined in [2]. If Λ is strictly dominant, by the last two propositions, we can define a group homomorphism Ad of K_ω^Λ into K_ω^{ad} such that

$$g\cdot x\cdot g^{-1}\cdot v=(\mathrm{Ad}\,(g)x)\cdot v \qquad \text{for any } g\in K_\omega^\Lambda,\ x\in H_1(\mathrm{ad}),\ v\in H_1(\Lambda).$$

Thus we get the adjoint representation of K_ω^Λ on $\mathfrak{g}_\omega$. Further Proposition 6.3 i) implies that each element in K_ω^{ad} defines an automorphism on the Lie algebra $\mathfrak{g}_\infty$ (resp. $\mathfrak{g}_\omega$) which leaves invariant the real subalgebra $\mathfrak{k}_\infty$ (resp. $\mathfrak{k}_\omega$). As a conclusion, we have obtained groups K_ω^{ad} and K_ω^Λ whose group structures are closely connected with the Lie algebra structure of $\mathfrak{k}_\omega$.

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