61. On Two Conjectures on Real Quadratic Fields

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Recently we learned from a paper of H. Yokoi [2] that there are two conjectures (C_1), (C_2) concerning the class numbers of real quadratic fields.

(C_1): Let l be a square-free integer of the form $l=q^2+4$ ($q \in N$). Then there exist just 6 quadratic fields $Q(\sqrt{l})$ of class number one.

(C_2): Let l be a square-free integer of the form $l=4q^2+1$ ($q \in N$). Then there exist just 6 quadratic fields $Q(\sqrt{l})$ of class number one.

In this paper, we shall prove that at least one of the two conjectures is true and that there are at most 7 quadratic fields $Q(\sqrt{l})$ of class number one for the other case. Our result will follow from two theorems which are independent of each other. Theorem 1 follows from Tatuzawa's lower bound for $L(1,\chi)$ [1], and Theorem 2 is obtained by the results of Yokoi [2] and by the help of a computer (Macsyma) in our Department.

In the sequel, l will always denote a square-free integer of the form $l=q^2+4$ or $l=4q^2+1$ ($q \in N$). We shall denote by $h(l)$ the class number of the quadratic field $Q(\sqrt{l})$.

Theorem 1. *There exists at most one $l \geqq e^{16}$ with $h(l)=1$.*

Proof. By Dirichlet's class number formula, we have

$$h(l)=\frac{\sqrt{l}}{2\log u}L(1,\chi_l),$$

where χ_l is the Kronecker character belonging to the quadratic field $Q(\sqrt{l})$ and u is the fundamental unit of $Q(\sqrt{l})$. By the choice of l, we have

$$u=\begin{cases}(q+\sqrt{l})/2 & \text{if } l=q^2+4,\\ 2q+\sqrt{l} & \text{if } l=4q^2+1.\end{cases}$$

Assume that $l \geqq e^{16}$. By Theorem 2 of [1], we have

$$L(1,\chi_l)>\frac{1}{16}(0.655)l^{-(1/16)}$$

with one possible exception of l.[1)]

Case 1. $l=q^2+4$. Then $u=(q+\sqrt{l})/2<\sqrt{l}$ and

$$h(l)=\frac{\sqrt{l}}{2\log u}L(1,\chi_l)>\frac{\sqrt{l}}{2\log\sqrt{l}}\,\frac{1}{16}(0.655)l^{-(1/16)}=\frac{1}{16}(0.655)\frac{l^{7/16}}{\log l}.$$

Since $f(x)=x^{7/16}/\log x$ is increasing on $[e^{16},\infty)$, we have

$$h(l)>\frac{1}{16}(0.655)\frac{e^7}{16}=2.805\cdots>2.$$

Case 2. $l=4q^2+1$. Then $u=2q+\sqrt{l}<2\sqrt{l}$ and

1) Put $k=l$ and $\varepsilon=1/16$ in Theorem 2 of [1].

$$h(l)=\frac{\sqrt{l}}{2\log u}L(1,\chi_l)>\frac{\sqrt{l}}{2\log 2+\log l}(0.655)\frac{1}{16}l^{-(1/16)}$$
$$=\frac{1}{16}(0.655)\frac{l^{7/16}}{2\log 2+\log l}.$$

Since $f(x)=x^{7/16}/(2\log 2+\log x)$ is increasing on $[e^{16},\infty)$,

$$h(l)>\frac{1}{16}(0.655)\frac{e^7}{2\log 2+16}>\frac{1}{16}(0.655)\frac{e^7}{20}=2.244\cdots>2.$$

This proves that $h(l)>2$ for all $l\geqq e^{16}$ except possibly one l, Q.E.D.

Theorem 2. *If* $h(l)=1$ *and* $l<e^{16}$, *then* $l=5, 13, 29, 53, 173, 293$ *for* $l=q^2+4$ *and* $l=5, 17, 37, 101, 197, 677$ *for* $l=4q^2+1$.

Proof. Since $h(l)=1$ by the assumption, l must be an odd prime and q must be prime or 1 (cf. Theorem 1 or [2]). In this situation, we have $(l/p)=-1$ for all odd primes $p<q$ (cf. Theorem 2 of [2]). But the tables below show that this is possible only for l's listed in the statement of the theorem, Q.E.D.

Conclusion. Since there is at most one exceptional l by Theorem 1 and $l=5$ is the only number common to numbers of the form $l=q^2+4$ and $l=4q^2+1$, we see from Theorem 2 that at least one of the conjectures (C_1), (C_2) is true and that there are at most 7 fields of class number one for the other case.

Remark. In the column p_0 of the tables the smallest odd prime $p_0<q$ such tnat $(l/p_0)=+1$ is given. These tables were obtained by the help of a computer (Macsyma) in our Department.

Table 1 ($l=q^2+4$)

q	l	p_0	q	l	p_0	q	l	p_0
1	5	—	293	85853	11	827	683933	13
3	13	—	307	94253	11	853	727613	13
5	29	—	313	97973	7	883	779693	19
7	53	—	317	100493	7	953	908213	11
13	173	—	347	120413	17	967	935093	11
17	293	—	373	139133	7	983	966293	11
37	1373	7	463	214373	11	997	994013	11
47	2213	7	487	237173	13	1087	1181573	7
67	4493	19	503	253013	17	1117	1247693	19
73	5333	11	547	299213	17	1123	1261133	11
97	9413	13	577	332933	13	1237	1530173	7
103	10613	7	593	351653	7	1367	1868693	7
137	18773	13	607	368453	7	1423	2024933	7
163	26573	7	613	375773	29	1447	2093813	7
167	27893	17	677	458333	7	1523	2319533	17
193	37253	23	743	552053	29	1543	2380853	19
233	54293	7	787	619373	13	1613	2601773	11
277	76733	19	823	677333	17	1627	2647133	11

Table 1 (Continued)

q	l	p_0	q	l	p_0	q	l	p_0
1637	2679773	29	2377	5650133	11	2843	8082653	23
1723	2968733	11	2477	6135533	13	2887	8334773	17
1753	3073013	11	2543	6466853	7	2903	8427413	7
1987	3948173	11	2633	6932693	11	2917	8508893	13
2003	4012013	11	2687	7219973	19	2957	8743853	13
2087	4355573	13	2693	7252253	7	3023	9138533	$> e^{16}$
2143	4592453	19	2777	7711733	7			
2333	5442893	7	2833	8025893	7			

Table 2 ($l=4q^2+1$)

q	l	p_0	q	l	p_0	q	l	p_0
1	5	—	193	148997	7	887	3147077	31
2	17	—	233	217157	17	947	3587237	11
3	37	—	317	401957	13	983	3865157	7
5	101	—	337	454277	19	1013	4104677	11
7	197	—	547	1196837	11	1063	4519877	17
13	677	—	587	1378277	13	1087	4726277	13
37	5477	13	647	1674437	7	1163	5410277	11
47	8837	11	653	1705637	19	1297	6728837	11
67	17957	7	677	1833317	19	1327	7043717	7
73	21317	7	683	1865957	7	1373	7540517	13
103	42437	23	773	2390117	7	1487	8844677	7
157	98597	7	827	2735717	13	1493	8916197	$> e^{16}$
163	106277	31	883	3118757	11			

Addendum. After we have written this paper, we learned that Chowla already conjectured (C_1). (cf. S. Chowla, *L*-series and elliptic curves, Number Theory Day, Lecture Notes, 626, Springer-Verlag, 1977, p. 2). Also Professor Iyanaga kindly communicated to us that *using the generalized Riemann hypothesis*, Mollin and Williams proved (C_1) and (C_2). (cf. R. A. Mollin, Class number one criteria for real quadratic fields. I, Proc. Japan Acad., 63A, 1987, pp. 121–125).

References

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[2] H. Yokoi: Class-number one problem for certain kind of real quadratic fields. Proc. International Conference on Class Numbers and Fundamental Units of Algebraic Number Fields, June 24–28, 1986, Katata, Japan, pp. 125–137.

[3] R. Sasaki: A characterization of certain real quadratic fields. Proc. Japan Acad., **62**A, 97–100 (1986).