59. The Irreducible Decomposition of the Unramified Principal Series

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1. Introduction. In what follows, we summarize some results on the irreducible decomposition of unramified principal series representations of quasi-split groups. A detailed account will be published elsewhere. In the case of split groups, the corresponding results were obtained by Rodier [3].

Let G be a connected reductive algebraic group defined over a non-archimedean local field F. We assume G is unramified, that is, G is quasi-split over F and split over an unramified extension of F. Let E be the minimal splitting field of G. Let S be a maximal F-split torus in G defined over F, T the centralizer of S in G, which is a maximal torus in G, and B a Borel subgroup of G defined over F containing T. We denote by $G(F)$, $B(F)$, $\cdots$, the locally compact and totally disconnected groups consisting of F-rational points of $G, B, \cdots$. Let $X^*(S)$ be the character group of S, $V=X^*(S)\otimes \boldsymbol{R}$ the vector space over the real number field $\boldsymbol{R}$ and Φ the relative root system of G with respect to S. A "root ray" of G with respect to S is an open half line with the starting point 0 in V containing at least one root relative to S. Let Ψ be the set of root rays of G with respect to S. For $a \in \Psi$, let $\sigma(a)$ (resp. $\tau(a)$) be the non-divisible (resp. non-multipliable) root contained in a. A root ray a is called *plural* if $\sigma(a)\neq\tau(a)$. We take the coroot system $\Psi^\vee$ attached to the reduced root system $\{\tau(a)\,|\,a \in \Psi\}$. The coroot corresponding to a root $\tau(a)$ is denoted by $a^\vee$. For $a \in \Psi$, we choose an absolute root α of G with respect to T such that the restriction of α to S equals $\sigma(a)$. Let Γ_α be the stabilizer of α in the Galois group Γ of E over F and $d(a)$ the index of Γ_α in Γ. Note that $d(a)$ is independent of the choice of α. Further, when a is a plural root ray, we put $\varepsilon(a)=(d(a)/2)+\pi(\log(q_F))^{-1}\sqrt{-1}$, where q_F is the cardinality of the residual field of F and $\pi=3.141\cdots$.

2. The unramified principal series. Let T_0 be the maximal compact subgroup of $T(F)$. An element of $X_0(T)=\mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$ is called an *unramified character* of $T(F)$. The relative Weyl group $W_G(S)$ corresponding to S acts on $X_0(T)$, namely, for $w \in W_G(S)$ and $x \in X_0(T)$, the action of w on χ is defined by $\chi^w(t)=\chi(\underline{w}^{-1}t\underline{w})$, $t \in T(F)$, where $\underline{w}$ is a representative of w in the group of F-rational points of the normalizer of S in G. An unramified character χ is called *regular* if $\chi^w\neq\chi$ for any $w \in W_G(S)$, $w\neq 1$. Let $X_{\mathrm{reg}}(T)$ be the set of regular unramified characters of $T(F)$. Note that each unramified character χ is trivially extended to a character of $B(F)$. Now, for $\chi \in X_0(T)$, the representation $I(\chi)$ of $G(F)$ induced by (the extension of) χ is by definition the right regular representation of $G(F)$ on the space of all locally constant functions $\phi: G(F) \to C$ such that $\phi(bg) = \delta_B(b)\chi(b)\phi(g)$ for any $b \in B(F)$, $g \in G(F)$, where δ_B is the positive square root of the modulus character of $B(F)$. $I(\chi)$ is called an *unramified principal series representation* of $G(F)$. It is known that, for any $\chi \in X_{\mathrm{reg}}(T)$, $I(\chi)$ has a unique irreducible subrepresentation of $G(F)$.

3. The irreducible decomposition of $I(\chi)$. We fix $\chi \in X_{\mathrm{reg}}(T)$ and denote by $H(\chi)$ the subset of $\Psi^\vee$ consisting of $a^\vee$, where a ($\in \Psi$) is non-plural and $\chi \circ a^\vee = |\cdot|_F^{d(a)}$ or a is plural and $\chi \circ a^\vee = |\cdot|_F^{d(a)}$ or $|\cdot|_F^{\varepsilon(a)}$, $|\cdot|_F$ denoting the normalized absolute value of F. Let $JH(\chi)$ be the set of irreducible constituents of $I(\chi)$ and $C(\chi)$ the set of connected components of $V - \bigcup_{a^\vee \in H(\chi)} \mathrm{Ker}\,(a^\vee)$. Note that the multiplicity one theorem holds for $I(\chi)$. Further, by a result of Bernstein and Zelevinsky [1], one may identify $JH(\chi)$ with $JH(\chi^w)$ for any $w \in W_G(S)$. For $D \in C(\chi)$, let $W(D) = \{w \in W_G(S) \mid w^{-1}C^+ \subset D\}$, where C^+ is the Weyl chamber in V corresponding to B. For $w \in W(D)$, let $\rho(D, w)$ be the unique irreducible subrepresentation of $I(\chi^w)$. Then, we have

Theorem 1. *Let $\chi \in X_{\mathrm{reg}}(T)$ and $D \in C(\chi)$. Then, for any $w_1, w_2 \in W(D)$, $\rho(D, w_1)$ is $G(F)$-isomorphic to $\rho(D, w_2)$. Namely, the isomorphism class of $\rho(D, w)$ $(w \in W(D))$ depends only on D.*

We write $\rho(D)$ for $\rho(D, w)$, considered as an element of $JH(\chi)$. Then we have a correspondence $\rho: C(\chi) \to JH(\chi)$, $D \mapsto \rho(D)$.

Theorem 2. *Let $\chi \in X_{\mathrm{reg}}(T)$.*

(1) *The map $\rho: C(\chi) \to JH(\chi)$ is bijective.*

(2) *Let $\langle H(\chi) \rangle$ be the set of coroots represented by an integral linear combination of elements of $H(\chi)$. Then, $\langle H(\chi) \rangle$ is a root system and $H(\chi)$ is a basis of $\langle H(\chi) \rangle$. In particular, the elements of $H(\chi)$ are linearly independent and the cardinality $|H(\chi)|$ of $H(\chi)$ is bounded by the semisimple F-rank of G.* (*Combining with* (1), *one sees that the cardinality of $JH(\chi)$ equals $2^{|H(\chi)|}$.*)

(3) *Let $D_\chi = \bigcap_{a^\vee \in H(\chi)} (a^\vee)^{-1}(\boldsymbol{R}_+)$, where $\boldsymbol{R}_+$ is the set of positive real numbers. Let φ be a non-degenerate character of the group $U(F)$ consisting of the F-rational points of the unipotent radical of B. Then, for $D \in C(\chi)$, $\rho(D)$ has a Whittaker model with respect to φ if and only if $D = D_\chi$.*

These theorems are proved by using Rodier's method ([3]) and the properties of irreducible root systems which appear as in constituents of $\Psi^\vee$.

References

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