7. *Commutator Relations in Kac-Moody Groups*

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In this note, we will calculate the commutator relations in Kac-Moody groups over commutative rings. The commutator relations have been discussed already in Tits [4]. Our approach is more elementary and more explicit.

1. Chevalley systems. Let A be an $n\times n$ generalized Cartan matrix, $\mathfrak{g}$ the associated Kac-Moody algebra over $\boldsymbol{C}$, being generated by the Cartan subalgebra $\mathfrak{h}$ and the Chevalley generators $e_1, \cdots, e_n, f_1, \cdots, f_n$, and Δ the root system of $(\mathfrak{g}, \mathfrak{h})$ with the standard fundamental system $\Pi=\{\alpha_1, \cdots, \alpha_n\}$. Then we obtain the root space decomposition $\mathfrak{g}=\oplus_{\alpha\in\Delta}\mathfrak{g}^\alpha$ with $\mathfrak{g}^0=\mathfrak{h}$, $\mathfrak{g}^{\alpha_i}=\boldsymbol{C}e_i$ and $\mathfrak{g}^{-\alpha_i}=\boldsymbol{C}f_i$ $(1\leq i\leq n)$. The Chevalley involution ω is defined to be the involutive automorphism of $\mathfrak{g}$ given by $\omega(e_i)=-f_i$, $\omega(f_i)=-e_i$, $\omega(h)=-h$ for all $1\leq i\leq n$ and $h\in\mathfrak{h}$. By the definition, $\omega(\mathfrak{g}^\alpha)=\mathfrak{g}^{-\alpha}$ for all $\alpha\in\Delta$ (cf. [3]).

For each $\alpha\in\Delta^{re}$, the set of real roots, a pair $(e_\alpha, e_{-\alpha})\in\mathfrak{g}^\alpha\times\mathfrak{g}^{-\alpha}$ is called a Chevalley pair for α if $[e_\alpha, e_{-\alpha}]=h_\alpha$ and $\omega(e_\alpha)+e_{-\alpha}=0$, where h_α is the coroot of α. There are precisely two Chevalley pairs for each $\alpha\in\Delta^{re}$. If one is $(e_\alpha, e_{-\alpha})$, then $(-e_\alpha, -e_{-\alpha})$ is the other. We choose and fix a Chevalley pair for each positive real root α with $e_{\alpha_i}=e_i$, $e_{-\alpha_i}=f_i$ $(1\leq i\leq n)$. Then the set $C=\{e_\alpha\,|\,\alpha\in\Delta^{re}\}$ is called a Chevalley system for Δ^{re}. Notice that $C\cup\{h_{\alpha_1}, \cdots, h_{\alpha_n}\}$ is a Chevalley basis of $\mathfrak{g}$ if A is of finite type (cf. [1], [2]).

Let $\alpha, \beta\in\Delta^{re}$. If $\alpha+\beta\in\Delta^{re}$, then we define the number $N_{\alpha\beta}$ by $[e_\alpha, e_\beta]=N_{\alpha\beta}e_{\alpha+\beta}$. Then we obtain the following result, which is useful for computing the commutator relations.

Theorem 1. *Let $\alpha, \beta\in\Delta^{re}$ with $\alpha+\beta\in\Delta^{re}$, and let $\beta-p\alpha, \cdots, \beta, \cdots, \beta+q\alpha$ $(p, q\in\boldsymbol{Z}_{\geq0})$ be the α-string through β. Then $N_{\alpha\beta}=\pm(p+1)$. In particular, $N_{\alpha\beta}\in\boldsymbol{Z}$.*

Proof. We can assume $n=2$, hence A is symmetrizable. We fix a symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ induced by A and having the property $(\alpha_i, \alpha_i)>0$. Then we see

$$N_{\alpha\beta}^2=(p+1)\left\{(p+1)-\left(2\frac{(\beta,\alpha)}{(\alpha,\alpha)}+1\right)\left(1-q\frac{(\alpha,\alpha)}{(\beta,\beta)}\right)\right\}$$

(cf. [2]). If $(\alpha,\alpha)\geq(\beta,\beta)$ and $(\beta,\alpha)<0$, then $\beta+i\alpha\in\Delta^{re}$ $(-p\leq i\leq q)$, and $p=0, 1$, hence $2(\beta,\alpha)/(\alpha,\alpha)=-1$. If $(\alpha,\alpha)\geq(\beta,\beta)$ and $(\beta,\alpha)\geq0$, then $q=1$ and $(\alpha+\beta, \alpha+\beta)>(\alpha,\alpha)=(\beta,\beta)$. If $(\alpha,\alpha)<(\beta,\beta)$, then $(\alpha,\beta)<0$ and $p=0$, hence $(\beta,\beta)/(\alpha,\alpha)=q$. Therefore, in any case, we obtain $N_{\alpha\beta}^2=(p+1)^2$ and $N_{\alpha\beta}=\pm(p+1)$. □

2. Commutator relations. Let $G(R)$ be a Kac-Moody group over a commutative ring R, with 1, of type A (cf. [4]). For each $\alpha \in \Delta^{re}$, there is a group homomorphism x_α of the additive group R^+ into $G(R)$. Indeed, we may write $x_\alpha(t) = \exp te_\alpha \in G(R)$. In $G(R)$, we want to calculate $[x_\alpha(s), x_\beta(t)]$ for $\alpha, \beta \in \Delta^{re}$ and $s, t \in R$. We will see that it is possible if

$$(*) \qquad (\boldsymbol{Z}_{\geq 0}\alpha + \boldsymbol{Z}_{\geq 0}\beta) \cap \Delta^{im} = \emptyset,$$

where $\Delta^{im} = \Delta - \Delta^{re}$, the set of imaginary roots, and obtain an explicit formula. To do this, we can assume $n=2$. We suppose the condition $(*)$ and $(\alpha, \alpha) \leq (\beta, \beta)$. Put $Q_{\alpha\beta} = (\boldsymbol{Z}_{\geq 0}\alpha + \boldsymbol{Z}_{\geq 0}\beta) \cap \Delta \subset \Delta^{re}$.

Theorem 2. *Notation and assumption are as above.*

(1) *If* $Q_{\alpha\beta} = \emptyset$, *then*
$$[x_\alpha(s), x_\beta(t)] = 1.$$

(2) *If* $Q_{\alpha\beta} = \{\alpha+\beta\}$, *then*
$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm(p+1)st).$$

(3) *If* $Q_{\alpha\beta} = \{\alpha+\beta,\ 2\alpha+\beta\}$, *then*
$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2 t).$$

(4) *If* $Q_{\alpha\beta} = \{\alpha+\beta,\ 2\alpha+\beta,\ \alpha+2\beta\}$, *then*
$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm 2st)x_{2\alpha+\beta}(\pm 3s^2 t)x_{\alpha+2\beta}(\pm 3st^2).$$

(5) *If* $Q_{\alpha\beta} = \{\alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta,\ 2\alpha+3\beta\}$, *then*
$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2 t)x_{3\alpha+\beta}(\pm s^3 t)x_{3\alpha+2\beta}(\pm 2s^3 t^2).$$

(*One of the five cases happens.*)

Remark. Theorem 1 and Theorem 2 are natural generalizations of the corresponding results in case of finite type. The essence is to reduce to the rank two case. Another sufficient condition to compute the above commutators has been given by J. Tits [4].

References

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