54. *Completely Integrable Symplectic Mapping*

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1. Introduction. The complete integrability, whose concept was first introduced by Liouville, is still an important subject in the recent development of Hamiltonian systems [1], [8]. This concept exists not only in Hamiltonian mechanical systems but also in certain systems of evolution equations. Also, the complete integrability can be found in symplectic mappings, which is the theme treated in this paper. Though any exponential map of a Hamiltonian vector field is a symplectic mapping, the converse is not true. Therefore, it is not always the case that a symplectic mapping possesses properties similar to those of a Hamiltonian flow. As far as complete integrability, however, similar features can be seen.

Any Hamiltonian version of a discrete-time mechanics is expressed in terms of a symplectic mapping [5], and the mechanics has also a Lagrangian formulation characterized by the discrete variational principle [4], [6]. Then, discrete systems expressed as the Euler equations, which appear in various branches, are transformed to symplectic mappings.

The contents of this paper are arranged as follows. A definition of a completely integrable symplectic mapping is given, and a discrete version of Liouville's theorem is proved. Furthermore, it is shown that its behavior is ergodic on (a submanifold of) a torus under some conditions.

2. Complete integrable symplectic map and its properties. In this section, we introduce the concept of complete integrable discrete mechanical systems and derive two important properties.

Let (M, ω) be a $2N$-dimensional symplectic manifold, ω being a symplectic structure. We think of a symplectic mapping ϕ on M. This is a Hamiltonian version of a discrete mechanical system. In addition to preservation of the symplectic structure, ϕ has a formal similarity to the usual canonical equations. That is, when ϕ is sufficiently near the identity mapping, it is expressed on any symplectic chart (Q^i, P_j) as

$$Q^i_{\tau+1}-Q^i_\tau=\frac{\partial H}{\partial P_i}(Q_\tau, P_{\tau+1}), \tag{1a}$$

$$P_{i,\tau+1}-P_{i,\tau}=-\frac{\partial H}{\partial Q^i}(Q_\tau, P_{\tau+1}), \tag{1b}$$

where τ takes the integral values representing the discrete time and H is a certain function [5].

A smooth function f on M is called an F.I. of ϕ, if and only if $\phi^* f=f$ holds, where ϕ^* denotes the pull-back associated with ϕ. The value of each F.I. remains constant along any solution of ϕ. Let us give the following definition.

Definition. A discrete mechanical system ϕ is called completely integrable, if and only if it admits N F.I.'s f_i such that

(1) $$\{f_i, f_j\}=0 \qquad (i, j=1, \cdots, N), \tag{2}$$

(2) $$df_1 \wedge df_2 \wedge \cdots \wedge df_N \neq 0. \tag{3}$$

It is Liouville that first introduced the notion of complete integrability into classical mechanics in order to obtain an analytic expression of a solution of a mechanical system. Our definition is the same as that of continuous systems except for the definition of F.I.'s. We start by proving the discrete version of Liouville's theorem.

We consider each f_i as a new momentum P_i. Since (2) holds, Lie and Carathéodory's theorem [3] assures that there are N functions Q^i such that

$$\{Q^i, P_j\}=\delta^i_j, \qquad \{Q^i, Q^j\}=0. \tag{4}$$

These Q^i are obtained by using a generating function of a canonical transformation, the function being constructed by quadrature. Due to (2) and (4), (Q^i, P_j) gives a symplectic coordinate chart on M. We note that ϕ is expressed as (1) on this chart. Since P_i are F.I.'s, it follows from (1b) that $\partial H/\partial Q^i=0$, and that H is a function of $P_{\tau+1}$ alone. Moreover, (1a) is reduced to $Q^i_{\tau+1}-Q^i_\tau=\text{const.}$ along any solution. Then, the summation procedure applied to this yields an analytic expression of solutions of ϕ:

$$Q^i_\tau=m^i\tau+n^i, \text{ (5a)} \qquad P_i=c_i, \text{ (5b)}$$

where m^i, n^i, and c_i are constants. Thus, we have proved the following theorem.

Theorem 1. *Suppose that a completely integrable discrete mechanical system is sufficiently near the identity mapping. Then, the analytic expression of its solutions can be obtained by quadrature.*

A completely integrable discrete system behaves ergodically under a certain condition. Let us define a level set of $(f_1, \cdots, f_N)$:

$$M_c=\{x \in M \mid f_i(x)=c_i, \ i=1, \cdots, N\},$$

where c_i are constants. When M_c is not void, it follows from (3) that M_c is an N-dimensional closed submanifold of M. The conditions (2) and (3) mean that a group R^N acts on M_c effectively. Furthermore, when M_c is compact and connected, the action of this group becomes transitive, and it follows that M_c is diffeomorphic to an N-dimensional torus T^N [2]. Also, it has been proved that there is a symplectic chart which contains M_c. Then, in this case, (5a) gives an expression of a solution of ϕ on the torus M_c.

Theorem 2. *Suppose that a level set M_c of a completely integrable discrete mechanical system ϕ is compact and connected. Then, each solution of ϕ is composed of a finite number of points, or is dense in M_c, or is dense on a submanifold of M_c. In the third case, the submanifold is diffeomorphic to a union of a finite number of less dimensional tori.*

Proof. We give the proof in the case of $N=2$, for a similar discussion is possible when $N\geqq 3$. Recall that T^2 is diffeomorphic to $S^1\times S^1$. Each Q^i in (5) is considered as a coordinate of S^1, and without loss of generality we can make its modulus one. Now, when both m^1 and m^2 are rational numbers, the solution of (5a) is composed of a finite number of points. When $\{1, m^1, m^2\}$ is linearly independent over the field of all rational numbers, it follows from Kronecker's theorem that the solution is dense in T^2. In the remaining case, we can assume that m^1 is irrational and $m^2=(s_1/t_1)m^1+s_2/t_2$, where (s_i, t_i) is a set of relatively prime integers such that $t_i\geqq 1$. When $s_i=0$, we put $t_i=1$. In this case, we have

$$Q^2(\tau)=\frac{s_1}{t_1}Q^1(\tau)+\frac{s_2}{t_2}\tau+\left(n^2-\frac{s_1}{t_1}n^1\right).$$

We fix an arbitrary integer n subject to $0\leqq n\leqq t_2-1$, and consider the subsequence of the solution of (5a) given by $\tau=t_2\nu+n$ $(\nu=0, 1, \cdots)$. This subsequence lies on a curve defined by

$$Q^2=\frac{s_1}{t_1}Q^1+\left(n^2-\frac{s_1}{t_1}n^1+\frac{s_2}{t_2}n\right),$$

where $Q^i+1\equiv Q^i$ is assumed. This curve C is a one-dimensional torus, for s_1/t_1 is a rational number. We can adopt Q^1 as a coordinate of C whose value is taken in $[0, t_1)$. The values $\{Q^1(t_2\nu+n)\}_{\nu=0,1,\dots}$ (mod t_1) form a dense subset of C, for m_1 is irrational and t_i are integers. Therefore, the subsequence is contained in C densely. This circumstance holds for an arbitrary n so that the proof is finished. □

3. Remark. As far as continuous Hamiltonian systems are concerned, it is already known that there are only few completely integrable ones [7], and the circumstance is probably true also in the discrete case. However, it is true that almost every linear symplectic mapping is completely integrable and the mutually commutative F.I.'s can be constructed from the mapping by means of an algebraic operation. Moreover, there are many systems which can be considered as systems perturbed from complete integrable ones. It seems that the results in this paper are connected with such a perturbed discrete system.

References

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