1. Super Oscillatory Integrals and a Path-integral for a Non-relativistic Spinning Particle

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Introduction. For a given 'Hamiltonian' described by even and odd Grassmann variables (called super Hamiltonian), we 'quantize' it by applying the method of the product integrals. Namely, introducing the supersymmetric version of the oscillatory integrals (super oscillatory integrals, for short) whose phase and amplitude functions are defined by a super Hamiltonian, we prove the convergence of its iterated integrals under suitable conditions by a similar procedure in Kitada [2]. Detailed proof will appear elsewhere.

Result. Let V_N be a vector space over $\boldsymbol{R}$ of dimension N with a positive definite inner product whose orthonormal basis is given by $\{e_j\}_{j=1}^N$ where $N=2l$. We denote by $Cl(V_N)$ the free algebra over $\boldsymbol{C}$ generated by 1 and $\{e_j\}_{j=1}^N$ with relations $e_je_k+e_ke_j=-2\delta_{jk}$ for $j, k=1, 2, \cdots, N$. We prepare another vector space V_{N+2} over $\boldsymbol{C}$ of dimension $N+2$ with a positive definite inner product whose orthonormal basis is given by $\{e_j\}_{j=-1}^N$. Assuming above relations hold for $j, k=-1, 0, 1, \cdots, N$, we define $Cl(V_{N+2})$ analogously as above. In $Cl(V_{N+2})$, putting $\sigma_j=(1/\sqrt{2})(e_{2j}+\sqrt{-1}e_{2j-1})$ and $\bar{\sigma}_j=(1/\sqrt{2})(e_{2j}-\sqrt{-1}e_{2j-1})$ for $j=0, 1, \cdots, l$, we get easily the following Grassmann relations $\sigma_j\sigma_k+\sigma_k\sigma_j=0$, $\bar{\sigma}_j\bar{\sigma}_k+\bar{\sigma}_k\bar{\sigma}_j=0$ and $\sigma_j\bar{\sigma}_k+\bar{\sigma}_k\sigma_j=2\sqrt{-1}\delta_{jk}$ for $j, k=0, 1, \cdots, l$. We denote by $\mathcal{G}r(l+1)$ the set of free algebra over $\boldsymbol{C}$ generated by 1 and $\{\sigma_j\}_{j=0}^l$. Let S be a set of elements of $\mathcal{G}r(l+1)$ represented as $\psi=\sum_{|a|:\text{even}}\psi_a\sigma^a$. Any element $\psi\in S$ is called a spinor. We consider a spin field $\psi=\psi(q)$ on $\boldsymbol{R}^n$, that is, ψ is a section of a bundle $\pi: \mathcal{S}=\boldsymbol{R}^n\times S\to\boldsymbol{R}^n$ represented as $\psi(q)=\sum_{|a|:\text{even}}\psi_a(q)\sigma^a$, for $q\in\boldsymbol{R}^n$. Denote by $\Gamma_0^\infty(\mathcal{S})$ a set of smooth sections on $\mathcal{S}$ with compact support. For $\psi\in\Gamma_0^\infty(\mathcal{S})$, we put $\|\psi\|^2=\sum_{|a|:\text{even}}\|\psi_a\|^2_{L^2(\boldsymbol{R}^n)}$. We denote by $L^2(\mathcal{S})(=\boldsymbol{H})$ the completion of $\Gamma_0^\infty(\mathcal{S})$ with respect to $\|\cdot\|$. Defining a super-space $\boldsymbol{R}^{n,l+1}$ as a set of points with even coordinates $x_1, x_2, \cdots, x_n$ and odd coordinates $\theta_0, \theta_1, \cdots, \theta_l$, we introduce function spaces over $\boldsymbol{R}^{n,l+1}$ as same as those over $\boldsymbol{R}^n$. We define a mapping $\sharp: \Gamma_0^\infty(\mathcal{S})\to C_{0,e}^\infty(\boldsymbol{R}^{n,l+1})$ by $(\sharp\psi)(x,\theta)\ (=f(x,\theta))=\sum_{|a|:\text{even}}\psi_a(x)\theta^a$, where $\psi_a(x)$ is the Grassmann extension of $\psi_a(q)$. Conversely, for any $f(x,\theta)\in C_{0,e}^\infty(\boldsymbol{R}^{n,l+1})$, we put $(\flat f)(q)=f(q,\sigma_0,\cdots,\sigma_l)$. As $\sharp\flat=\mathrm{Id}$ and $\flat\sharp=\mathrm{Id}$, we have a natural identification between $\Gamma_0^\infty(\mathcal{S})$ (resp. $L^2(\mathcal{S})$) and $C_{0,e}^\infty(\boldsymbol{R}^{n,l+1})$ (resp. $L_e^2(\boldsymbol{R}^{n,l+1})$). Now, we may define an action ρ of $Cl(V_N)$ on $\mathcal{S}$ as

$$\rho(e_{2j})=(1/2)(\sigma_0+\sqrt{-1}*\bar{\sigma}_0\rfloor)(\sigma_{2j}+\sqrt{-1}*\bar{\sigma}_{2j}\rfloor)$$

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and

$$\rho(e_{2j-1})=(1/2)(\sigma_0+\sqrt{-1}*\bar{\sigma}_0 \rfloor)(\sigma_{2j}-\sqrt{-1}*\bar{\sigma}_{2j} \rfloor) \qquad \text{for } j=1, 2, \cdots, l.$$

Here $*$ means the complex conjugation of the coefficients in ψ and $\rfloor$ stands for the inner product on $\mathcal{S}$ viewed as an exterior algebra. Moreover, above defined action extended as a representation ρ of $Cl(V_N)$ on $\mathcal{S}$ may be expressed as even differential operators on $\boldsymbol{R}^{n,l+1}$, i.e.

$$\sharp\rho(e_{2j})\flat=\frac{1}{2}\left(\theta_0+\sqrt{-1}\frac{\partial}{\partial\theta_0}\right)\left(\theta_j+\sqrt{-1}\frac{\partial}{\partial\theta_j}\right)$$

and

$$\sharp\rho(e_{2j-1})\flat=\frac{1}{2\sqrt{-1}}\left(\theta_0+\sqrt{-1}\frac{\partial}{\partial\theta_0}\right)\left(\theta_j-\sqrt{-1}\frac{\partial}{\partial\theta_j}\right) \qquad \text{for } j=1, 2, \cdots, l.$$

Concerning the differential and the integral calculus on superspace, see, for example, Vladimirov and Volovich [3, 4].

Now, consider a super Hamiltonian $H(x;\xi,\theta;\pi)$ defined on $T^*\boldsymbol{R}^{n,l+1}$ with the following conditions: (A.1) $H(x;\xi,\theta;\pi)\in C_e^\infty(T^*\boldsymbol{R}^{n,l+1})$, (A.2) $H(x_B;\xi_B,0;0)$ is a smooth real valued function on $T^*(\boldsymbol{R}^n)$. (A.3) For any multi-indices a, b, α and β with $|\alpha|+|\beta|+|a|+|b|\geqq 2$, there exists a positive constant $C_{a,b,\alpha,\beta}$ such that

$$|\partial_{x_B}^\alpha\partial_{\xi_B}^\beta\vec{\partial}_\theta^a\vec{\partial}_\pi^b H(x_B;\xi_B,0;0)|\leqq C_{a,b,\alpha,\beta}<\infty.$$

We denote a solution of the following equations as $(x(t);\xi(t),\theta(t);\pi(t))$:

$$\frac{d}{dt}x=-\partial_\xi H,\quad \frac{d}{dt}\xi=-\partial_x H,\quad \frac{d}{dt}\theta=-\vec{\partial}_\pi H,\quad \frac{d}{dt}\pi=-\vec{\partial}^\theta H$$

satisfying the initial condition $(x(s);\xi(s),\theta(s);\pi(s))=(y;\eta,\omega;\rho)\in T^*\boldsymbol{R}^{n,l-1}$. Here $\vec{\partial}$ means the left derivative with respect to odd variables. If it is necessary to make explicit the dependence on the initial data, we rewrite $x(t)$ as $x(t,s,y;\eta,\omega;\rho)$ etc. Then for sufficiently small $\delta>0$ and fixed (η,ρ), a mapping

$$(y,\omega)\to(x(t,s,y;\eta,\omega;\rho),\ \theta(t,s,y;\eta,\omega;\rho))$$

is a global diffeomorphism from $\boldsymbol{R}^{n,l+1}$ to $\boldsymbol{R}^{n,l+1}$ for $|t-s|<\delta$. From this, we may define a mapping from $\boldsymbol{R}^{n,l+1}$ to $\boldsymbol{R}^{n,l+1}$ by

$$(x,\theta)\to(y(t,s,x;\eta,\theta;\rho),\ \omega(t,s,x;\eta,\theta;\rho)).$$

Putting $\langle\eta|y\rangle=\sum_{j=1}^n\eta_jy_j$ and $\langle\rho|\omega\rangle=\sum_{r=0}^l\rho_r\omega_r$, we introduce a Lagrangean function as

$$L(x;\xi,\theta;\pi)=\langle\xi|\partial_\xi H(x;\xi,\theta;\pi)\rangle+\langle\pi|\vec{\partial}_\pi H(x;\xi,\theta;\pi)\rangle-H(x;\xi,\theta;\pi).$$

Defining

$$g(t,s,y;\eta,\omega;\rho)=\langle\eta|y\rangle+\langle\rho|\omega\rangle+\int_s^t L(x(\tau);\xi(\tau),\theta(\tau);\pi(\tau))d\tau,$$

we introduce

$$\phi(t,s,x;\xi,\theta;\pi)=g(t,s,y(t,s,x;\xi,\theta;\pi);\xi,\omega(t,s,x;\xi,\theta;\pi);\pi).$$

For $|t-s|<\delta$, we consider the following transformation acting on $C_0^\infty(\boldsymbol{R}^{n,l+1})$:

$$\tilde{E}(t,s)u(x,\theta)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^{n,l+1}}\exp(i\phi(t,s,x;\xi,\theta;\pi))(Fu)(\xi,\rho)d\xi d\pi$$

where Fu is the Fourier transformation on $\boldsymbol{R}^{n,l+1}$ given by

$$Fu(\xi,\pi)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^{n,l+1}}\exp(-i\langle\xi|y\rangle-i\langle\pi|\omega\rangle)u(y,\omega)dyd\omega.$$

Combining these, we define a linear operator acting on $\Gamma_0^\infty(\mathcal{S})$ by $E(t)\psi(q) = \flat\tilde{E}(t)\sharp\psi(q)$, which may be extended to a bounded linear operator acting on $\boldsymbol{H}$.

Fix $T>0$. Let $[s, t]$ be an arbitrary given interval in $(-T, T)$ and let it be decomposed as $\Delta : s=t_0<t_1<\cdots<t_L=t$. Putting $\delta(\Delta)=\max_{1\leq j\leq L}|t_j-t_{j-1}|$, we introduce the product of integral operators $E(\Delta|t, s)$ attached to the above subdivision by $E(\Delta|t, s)=E(t, t_{L-1})\cdots E(t_1, s)$.

Theorem. *Under assumptions* (A.1)–(A.3), $E(\Delta|t, s)$ *defined as above converges to a linear bounded operator* $U(t, s)$ *on* $\boldsymbol{H}$ *when* $\delta(\Delta)\to 0$ *in the uniform operator topology. That is, for any subdivision* Δ *of* $[s, t]$ *such that* $\delta(\Delta)$ *is sufficiently small, we have*

$$\|U(t, s)-E(\Delta|t, s)\|\leqq C_1|t-s|\exp(C_1|t-s|/2)\delta(\Delta).$$

Here, C_1 *is some positive constant independent of* Δ, $s, t\in(-T, T)$ *and* $\|\cdot\|$ *stands for the operator norm in* $\boldsymbol{H}$. *Moreover, the family of bounded operators* $\{U(t, s)\,|\,t, s\in(-T, T)\}$ *satisfies the following properties*: (i) $U(s, s)=\mathrm{Id}$. (ii) *For any* $\psi\in\boldsymbol{H}$, *the mapping* $t\in(-T, T)\to U(t, s)\psi\in\boldsymbol{H}$ *is continuous.* (iii) $U(t_3, t_2)U(t_2, t_1)=U(t_3, t_1)$ *for any* $t_i\in(-T, T)$. (iv) *For any* $t, s\in(-T, T)$ *and* $\psi\in\Gamma_0^\infty(\mathcal{S})$, $U(t, s)\psi$ *is differentiable with respect to* t *and it satisfies*

$$\frac{d}{dt}U(t, s)\psi+i\bar{H}U(t, s)\psi=0.$$

Here $\bar{H}$ *is given by* $\bar{H}=\flat H\sharp$ *where*

$$Hu(x, \theta)=\int_{\boldsymbol{R}^{n,l+1}} H(x;\xi,\theta;\pi)\exp(-i\langle\xi|x\rangle-i\langle\pi|\theta\rangle)(Fu)(\xi,\pi)d\xi d\pi.$$

Remark. Putting $n=N$ and using the representation $\sharp\rho(\cdot)\flat$ defined before, we get a system of pseudo-differential operators of order less than 2 on $\boldsymbol{R}^{N,l+1}$ developed in Getzler [1].

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