18. On the Heat Operators of Cuspidally Stratified Riemannian Spaces

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Introduction and statement of the Main Theorem. In this paper, we intend to examine a property of the Laplacian Δ of generalized Neumann or Dirichlet types ([2]) acting on the space of square-integrable forms on certain *incomplete* Riemannian manifolds called *cuspidally stratified Riemannian spaces* (briefly, *CSR-spaces*).

Main Theorem. *The heat operator $e^{-\Delta t}$ on a real n-dimensional CSR-space is of trace class and there exists a constant $K>0$ such that*

$$\mathrm{Tr}\, e^{-\Delta t} \leqq K t^{-n/2}, \qquad 0<t\leqq t_0.$$

The author's study of CSR-spaces was motivated by the desire to prove a similar result for the smooth part $\mathcal{X}$ of a projective variety X (with the induced Fubini-Study metric, which is therefore incomplete). However such spaces do not fall into the category of CSR-spaces studied in this paper. In fact, even on a normal singular projective surface, the metric near the singular point is more complicated ([3]). Nevertheless the author believes that it will not be too difficult to extend the theory of CSR-spaces to projective varieties and that this will provide a suitable framework for studying the global analysis of singular projective varieties.

§ 1. Definition of CSR-spaces. Let X be a real n-dimensional compact stratified space (possibly with boundary) with Thom structure $\{\mathcal{T}, \mathcal{S}\}$ ([4]). Here $\mathcal{S}$ is the stratification of X (that is, a decomposition of X into smooth manifolds without boundaries) and $\mathcal{T}$ is a collection of open tubular neighborhoods of the strata (i.e., the elements of $\mathcal{S}$), where each open tubular neighborhood T_V ($V\in\mathcal{S}$) is endowed with the following three objects: the structure of a fibre bundle, $\pi_V : T_V\to V$, a so-called distance function from V, $\lambda_V : T_V\to[0,\infty)$, and a homeomorphism h_V from the mapping cylinder $M(\pi_V\,|\,\lambda_V^{-1}(1))$ to T_V. Note that $(T_V, \pi_V, \lambda_V, h_V)$, $V\in\mathcal{S}$, are compatible with each other in a natural sense.

Now let Σ be the (disjoint) union of the strata with positive codimensions and set $\mathcal{X}=X-\Sigma$. This manifold together with the metric g described below is called a *CSR-space*.

For each stratum $V\in\mathcal{S}$ with $\dim V<n$, let k_V be a real number with $k_V=0$ if $\dim V=n-1$ and $k_V\geqq 1$ if $\dim V<n-1$; set $\boldsymbol{k}=\{k_V : V\in\mathcal{S}, \dim V<n\}$. Then the metric g depends on $\boldsymbol{k}$ and is characterized near the strata with positive codimensions as follows:

For any $V \in \mathcal{S}$ with $\dim V < n$ and $x \in V$, set $\mathcal{X}_{V,x} = \pi_V^{-1}(x) \cap \mathcal{X}$ and $\dot{\mathcal{X}}_{V,x} = \lambda_V^{-1}(1) \cap \mathcal{X}_{V,x}$. Then we can identify $\mathcal{X}_{V,x}$ with $(0, \infty) \times \dot{\mathcal{X}}_{V,x}$ by the homeomorphism h_V. Hence the intersection of $\mathcal{X}$ and a (particular) neighborhood of x in X can be canonically identified (using the structure of T_V) with

$$U \times (0, 1) \times \dot{\mathcal{X}}_{V,x}, \tag{1.1}$$

where U is a neighborhood of x in V. Now fix a metric $\tilde{g}_V$ on V and let $\tilde{g}_{V|U}$ be its restriction to U. Let $dr \otimes dr$ be the standard metric on the interval $(0, 1)$ and $\dot{g}_{V,x}$ be the restriction of the given metric g to $\dot{\mathcal{X}}_{V,x}$. Then, on the manifold (1.1), the given metric g is quasi-isometric (by the identity map) to the metric

$$\tilde{g}_{V|U} + dr \otimes dr + r^{2k_V} \dot{g}_{V,x}. \tag{1.2}$$

Recall that the diffeomorphism $f : (\mathcal{X}_1, g_1) \to (\mathcal{X}_2, g_2)$ is called a quasi-isometry if there exists a constant $C > 0$ such that $C^{-1} g_1 \leqq f^* g_2 \leqq C g_1$; therefore, if the $\mathcal{X}_j$ are compact, then the diffeomorphism f is always quasi-isometric. Hence the above characterization is very rough. In fact, for example, the metric g does not even have to be divided into the form (1.2).

§ 2. **Idea of the proof of the Main Theorem.** The proof closely follows the program given by J. Cheeger ([1], which, in the notation above, has only treated $(\mathcal{X}, g)$ with $k = \{k_V = 1 : V \in \mathcal{S}, \dim V < n-1\}$). We have only to prove the following: let $(Y, \tilde{g})$ be an m-dimensional Riemannian manifold which has the property mentioned in Main Theorem (briefly, has the *property* (MT)) and set

(2.1) $C^{\rho}_{<R}(Y) =$ "the space $(0, R) \times Y$ with the metric $dr \otimes dr + \rho(r)^2 \tilde{g}$",

where $\rho(r) = r^k$ for a number $k \geqq 1$; then this *metric cusp* has the property (MT).

Now start by finding the system of fundamental solutions of the differential equation $\Delta\theta = \lambda\theta$, $\lambda > 0$, by using the method of the separation of variables (in the r- and Y- directions; in the Y-direction the inductive assumption requires the possibility of the series expansions of square-integrable forms in terms of eigenforms). Then the spectrum of the following singular boundary value problem on the interval $(0, R]$ turns out to be the non-trivial part of the spectrum of the Laplacian on (2.1). (The remaining part, the trivial part, comes from the zero, maximal and minimal points of the Neumann and Bessel functions.)

$$\begin{aligned} &H''(r) + \{\lambda - q_\mu(r)\} H(r) = 0, \qquad 0 < r \leqq R, \\ &\int_0^R H(r)^2 dr < \infty, \\ &\frac{d}{dr}(\rho^{-\kappa/2} H)(R) = 0 \qquad (\text{or } (\rho^{-\kappa/2} H)(R) = 0), \\ &q_\mu(r) = \mu r^{-2k} + \frac{k\kappa(k\kappa - 2)}{4} r^{-2} \quad (>0). \end{aligned} \tag{2.2}$$

Here μ belongs to the positive spectrum of the Laplacian on Y $(\sqrt{\mu + ((1-\kappa)/2)^2} \geqq 1$ if $k = 1)$, which is discrete, and the constant κ is determined by the dimension m of Y and the degree of forms we are considering. The general expansion theorem ([6]) says that the spectrum of the problem (2.2) consists of increasing eigenvalues, $(q_\mu(R)<)\lambda_1(\mu)<\lambda_2(\mu)<\cdots\uparrow\infty$, and the proper comparison theorem implies the existence of a constant $K>0$ which is independent of $\mu>0$, such that

$$\sum_{j=1} e^{-\lambda_j(\mu)t} \leqq K t^{-1/2} e^{-q_\mu(R)t}, \qquad 0<t\leqq t_0. \tag{2.3}$$

Hence, by using the property (MT) of Y, we get

Lemma. *There exists a constant $K>0$ such that*

$$\sum_{\mu>0}\sum_{j=1}^{\infty} e^{-\lambda_j(\mu)t} \leqq K t^{-(m+1)/2}, \qquad 0<t\leqq t_0. \tag{2.4}$$

This essentially proves that the metric cusp (2.1) has the property (MT).

References

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