14. *Interpolation of Linear Operators in Lebesgue Spaces with Mixed Norm*

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The aim of this paper is to show that a bounded linear operator in the Lebesgue spaces $L^t(M^n; L^s(M^m))$ with mixed norm is bounded in the space $L^u(M^{m+n})$ under a condition on (s, t), where $1/u=(m/s+n/t)/(m+n)$. As applications we shall have a result on Riesz-Bochner summing operator and on the restriction problem of Fourier transform.

1. Notations. Let $(M, \mathcal{M}, \mu)$ and $(N, \mathcal{N}, \nu)$ be σ-finite measure spaces, and $(M_j, \mathcal{M}_j, \mu_j)$ $(j=0, 1, \cdots)$ be copies of $(M, \mathcal{M}, \mu)$. Let $d\geq 2$ and $(\bar{M}, \bar{\mathcal{M}}, \bar{\mu})$ be the product measure space $\prod_{j=0}^{d-1}(M_j, \mathcal{M}_j, \mu_j)$. For a subset $p=\{p_0, p_1, \cdots, p_{m-1}\}\subset\{0, 1, \cdots, d-1\}$ put

$$(M(p), \mathcal{M}(p), \mu(p))=\prod_{j\in p}(M_j, \mathcal{M}_j, \mu_j).$$

Thus

$$d\mu(p)(x_{p_0}, \cdots, x_{p_{m-1}})=d\mu_{p_0}(x_{p_0})\cdots d\mu_{p_{m-1}}(x_{p_{m-1}}) \quad \text{and} \quad d\bar{\mu}=d\mu(p)\times d\mu(p'),$$

where p' denotes the complement $\{0, 1, \cdots, d-1\}\backslash p$. $(\bar{N}, \bar{\mathcal{N}}, \bar{\nu})$ and $(N(p), \mathcal{N}(p), \nu(p))$ will be defined similarly.

Let $1\leq s, t<\infty$. $L^s(\bar{M})$ denotes the Lebesgue space with norm $\|f\|_s=\left(\int_{\bar{M}}|f|^s d\bar{\mu}\right)^{1/s}$ and $L^t(L^s)=L^t(M(p'); L^s(M(p)))$ the Lebesgue space with mixed norm

$$\|f\|_{(t,s;p)}=\left[\int_{M(p')}\left(\int_{M(p)}|f|^s d\mu(p)\right)^{t/s}d\mu(p')\right]^{1/t}.$$

The definition for the cases $s=\infty$ and/or $t=\infty$ will be modified obviously.

Let m and n be positive integers such that $d=m+n$. We define $u\geq 1$ by

$$1/u=(m/s+n/t)/d.$$

For $1\leq s\leq\infty$, s' will denote the conjugate exponent $s/(s-1)$.

P denotes the family $\{p\in\{0, 1, \cdots, d-1\}; \operatorname{card}(p)=m\}$ if $m\geq n$ and $P=\{0, 1, \cdots, d-1\}$ otherwise. Let $\{I_p; p\in P\}$ be a family of disjoint arcs in the unit circle of length $2\pi/\operatorname{card}(P)$.

2. Theorems.

Lemma 1. *Assume $1\leq s\leq t\leq\infty$. Let w and f be simple functions in $(\bar{M}, \bar{\mathcal{M}}, \bar{\mu})$. Then there exist functions $W^z(x)$ and $F^z(x)$ on $\bar{M}$ such that*

(i) *$W^z(x)$ and $F^z(x)$ are bounded and holomorphic in $|z|<1$ for every $x\in\bar{M}$, and measurable in x for every $|z|<1$,*

(ii) *$W^0(x)=w(x)$ and $F^0(x)=f(x)$,*

(iii) *$\|W^z\|_{(t,s;p)}\leq\|w\|_u$ for $z\in\operatorname{int}(I_p)$ and $p\in P$,*

and

(iv) *furthermore if f is of the form $f_0(x_0)f_1(x_1)\cdots f_{d-1}(x_{d-1})$, then*

$$\|F^z\|_{(t',s';p)}\leqq\|f\|_{u'}.$$

As an easy consequence of Lemma we get the followings.

Theorem 1. *Let T be a linear operator of simple functions on $(\bar{M}, \bar{\mathcal{M}}, \bar{\mu})$ to measurable functions on $(N, \mathcal{N}, \nu)$. Let $v(e^{i\theta})$ be a measurable function such that $1\le v(e^{i\theta})\le\infty$ and*

$$1/v=\int_0^{2\pi} 1/v(e^{i\theta})\frac{d\theta}{2\pi}.$$

Let $1\le u_0\le u_1\le\infty$ and

$$1/u=(m/u_0+n/u_1)/d.$$

Suppose

$$\|Tw\|_{v(e^{i\theta})}\le C(e^{i\theta})\|w\|_{(u_1,u_0;p)}$$

for all simple functions w, $\theta\in\mathrm{Int}\,(I_p)$ and $p\in P$, where $C(e^{i\theta})$ is measurable. Then

$$\|Tw\|_v\le C\|w\|_u,$$

with

$$C=\exp\left(\int_0^{2\pi}\log C(e^{i\theta})\frac{d\theta}{2\pi}\right).$$

Theorem 2. *Let T be a linear operator of simple functions on $\bar{M}$ to measurable functions on $\bar{N}$. Let $1\le u_0\le u_1\le\infty$ and $1\le v_1\le v_0\le\infty$. Suppose that*

$$\|Tw\|_{(v_1,v_0;p)}\le C_p\|w\|_{(u_1,u_0;p)}$$

for all w and $p\in P$.

If

$$1/u=(m/u_0+n/u_1)/d \quad \text{and} \quad 1/v=(m/v_0+n/v_1)/d,$$

then

$$\|Tw\|_v\le C\|w\|_u,$$

where

$$C=\Big(\prod_{p\in P} C_p\Big)^{1/\mathrm{card}(P)}.$$

Theorem 3. *Let T be a linear operator of simple functions on $\bar{M}$ to measurable functions on $\bar{N}$. Let $1\le u_1\le u_0\le\infty$ and $1\le v_1\le v_0\le\infty$. Suppose that*

$$\|Tf\|_{(v_1,v_0;p)}\le C(p)\|f\|_{(u_1,u_0;p)}$$

for all simple functions f of the form $f_0(x_0)\cdots f_{d-1}(x_{d-1})$ and $p\in P$. Then

$$\|Tf\|_v\le C\|f\|_u$$

for all f of the product form, where u, v and C are defined in Theorem 2.

Remark 1. Suppose $1\le u_1\le u_0\le\infty$. Then the conclusion of Theorem 1 holds for w of the form $w_0(x_0)w_1(x_1)\cdots w_{d-1}(x_{d-1})$, but in general, it does not hold.

Remark 2. The family of the spaces $L^{v(e^{i\theta})}(N)$ in Theorem 1 is replaced by the more general family of Banach spaces $B[z]$ introduced by [1].

3. Applications. For a reasonable function f on the d-dimensional Euclidean space R^d the Riesz-Bochner operator $s^\varepsilon(f)$ of order ε is defined by $s^{\varepsilon}(f)^{\wedge}(\xi)=(1-|\xi|^2)_+^{\varepsilon}\hat{f}(\xi)$, where $\hat{f}$ is the Fourier transform of f and $a_+=\max(0, a)$. Let P be the family with $m=d-2$ and we use the notations in § 1 with $M=N=R^1$.

Theorem 4. *Let $\varepsilon>0$. Then*

$$\|s^{\varepsilon}(f)\|_{(4,2;p)}\leq C\|f\|_{(4,2;p)}$$

for all $p\in P$ and f, where C is a constant independent of f.

Applying Theorem 3 to Theorem 4 we get

Theorem 5. *Let $\varepsilon>0$. Then*

$$\|s^{\varepsilon}(f)\|_{2d/(d+1)}\leq C\|f\|_{2d/(d+1)}$$

for all f of the form $f_0(x_0)f_1(x_1)\cdots f_{d-1}(x_{d-1})$.

Another application is the following.

Theorem 6. *If f is a continuous function of the form $f_0(x_0)f_1(x_1)\cdots f_{d-1}(x_{d-1})$ with compact support, then*

$$\left[\int_{S^{d-1}}|\hat{f}(\xi)|^2|\xi_0\cdots\xi_{d-1}|^{1/d}\,d\sigma(\xi)\right]^{1/2}\leq C\|f\|_{2d/(d+1)}$$

with a constant independent of f.

A detailed proof of the theorems will be published elsewhere.

Reference

[1] R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss: A theory of complex interpolation for families of Banach spaces. Adv. in Math., **43**, 203–229 (1982).