13. On the Connection of the White-Noise and Malliavin Calculi

By Jürgen POTTHOFF*)
Department of Mathematics, Nagoya University
Department of Mathematics, Technical University, Berlin

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1986)

0. Introduction. In this note we show how the basics of the Malliavin calculus, see e.g. [5, 6], can be formulated in the frame work of Hida's white-noise calculus [1, 2, 4].

The original motivation of Malliavin to introduce his calculus was to prove statements about the distributions generated by Wiener-functionals, particularly whether these distributions are absolutely continuous. It turns out that his method can be expressed in a rather simple manner by the white-noise calculus. Only basic formulae are needed, such as the chain rule, integration by parts for the ∂_t derivatives and the product rule for ∂_t^*.

Throughout this note we adopt the notation of Kuo [4], however we shall use the definition of the ∂_t-operator given in [3]; for general background see also [1].

1. The chain rule. In this section we establish the chain rule for ∂_t.

Let $(\mathcal{S}'(\boldsymbol{R}), \Sigma, d\mu)$ be white noise and consider a functional φ on $\mathcal{S}'(\boldsymbol{R})$. For fixed $x \in \mathcal{S}'(\boldsymbol{R})$, let φ_x be the functional on $\mathcal{S}(\boldsymbol{R})$ defined by $\varphi_x(\xi)=\varphi(x+\xi)$, $\xi \in \mathcal{S}(\boldsymbol{R})$.

Proposition. *Let $\varphi \in L^p(d\mu)$, $p>1$, so that $\varphi_x(\xi)$ and $\int \varphi_x(\xi)d\mu(x)$ are Fréchet-differentiable on $\mathcal{S}(\boldsymbol{R})$. Then*

$$\frac{\delta}{\delta\xi(t)} \int \varphi_x(\xi)d\mu(x)=\int \frac{\delta}{\delta\xi(t)}\varphi_x(\xi)d\mu(x). \tag{1.1}$$

Corollary.

$$(\partial_t\varphi)(\xi+x)=\left(\frac{\delta}{\delta\xi(t)}\varphi_x\right)(\xi). \tag{1.2}$$

Sketch of proof. (1.1) follows by use of Gâteaux-derivatives and the dominated convergence theorem; the additional use of the chain rule for Fréchet-derivatives gives (1.2).

Lemma (chain rule). *If $\varphi=(\varphi_1, \cdots, \varphi_d)$ is an $\boldsymbol{R}^d$-valued $\mathcal{S}'(\boldsymbol{R})$-functional, with each φ_i satisfying the assumptions of the proposition and $F \in C^1(\boldsymbol{R}^d, \boldsymbol{R})$, so that $F \circ \varphi \in L^q(d\mu)$, $q>1$, then*

$$\partial_t F \circ \varphi=\sum_{i=1}^{d} (F_{,i} \circ \varphi)\partial_t\varphi_i. \tag{1.3}$$

Here $F_{,i}$ denotes the i^{th} partial derivative of F.

*) Supported by JSPS.

This lemma follows easily from (1.1), (1.2) and the chain rule for Fréchet-derivatives.

2. A link between white-noise and Malliavin calculi. Expressing the Wiener process $B(t)$ at time t as $\langle x, 1_{[0,t)}\rangle$, $x \in \mathcal{S}'(\boldsymbol{R})$, [1, 4], we consider a Wiener-functional ϕ as an $\mathcal{S}'(\boldsymbol{R})$-functional φ, $\varphi(x)=\phi(\langle x, 1_{[0,\cdot)}\rangle)$.

Let ϕ (and hence φ) take values in $\boldsymbol{R}^d$, φ satisfying the same hypothesis as in section 1, let $F \in \mathcal{D}(\boldsymbol{R}^d)$. Consider

$$I_l := \int (F_{,l} \circ \varphi)(x)\psi(x)d\mu(x), \qquad 1 \leq l \leq d, \tag{2.1}$$

ψ some $\mathcal{S}'(\boldsymbol{R})$-functional. According to Malliavin [5] (see also [6]) we are interested in a bound of the form $|I_l| \leq \text{const.}\ \|F\|_\infty$.

Define the white-noise functional

$$\langle\!\langle \varphi_i, \varphi_j \rangle\!\rangle(x) := \int_{\boldsymbol{R}} (\partial_t \varphi_i)(x)(\partial_t \varphi_j)(x)dt, \qquad 1 \leq i,\ j \leq d. \tag{2.2}$$

Suppose that the matrix $(\langle\!\langle \varphi_i, \varphi_j \rangle\!\rangle(x))$ has (μ-a.e.) an inverse $\gamma(x)$. Then we can invert (1.3):

$$(F_{,l} \circ \varphi)(x) = \sum_{i=1}^{d} \int (\partial_t F \circ \varphi)(x)\gamma_{li}(x)(\partial_t \varphi_i)(x)dt \qquad (\mu\text{-a.e.}). \tag{2.3}$$

Inserting (2.3) into (2.1), using Fubini's theorem and integration by parts yields

$$I_l = \int F \circ \varphi \int \partial_t^* \psi \sum_{i=1}^{d} \gamma_{li} \partial_t \varphi_i dt d\mu. \tag{2.4}$$

Using the product rule for ∂_t^* [3], we find

$$I_l = \int (F \circ \varphi)(x)\Gamma_l(x)\, d\mu(x) \tag{2.5.a}$$

with

$$\Gamma_l(x) = \sum_{i=1}^{d} \left\{ \psi(x)\gamma_{li}(x) \int \partial_t^* \partial_t \varphi_i(x)dt - \gamma_{li}(x)\langle\!\langle \psi, \varphi_i \rangle\!\rangle(x) - \psi(x)\langle\!\langle \gamma_{li}, \varphi_i \rangle\!\rangle(x) \right\}. \tag{2.5.b}$$

This is the analogue of the basic formula of the Malliavin calculus in the white-noise language. With the lemma in I.1 of [5] we have the following (setting $\psi \equiv 1$).

Theorem. *Let φ be associated with the Wiener functional ϕ, φ as before. Assume that γ exists μ-a.e. and that $\Gamma_l \in L^1(d\mu)$, $1 \leq l \leq d$. Then the distribution of ϕ on $\boldsymbol{R}^d$ is absolutely continuous w.r.t. Lebesgue measure.*

Iterations of (2.5) for higher partial derivatives of F, provide information on the differentiability of the density of ϕ.

In the applications the crucial point is to prove the invertibility of $(\langle\!\langle \varphi_i, \varphi_j \rangle\!\rangle)$. To study this problem, Stroock derives identities for this expression in [6], in case that ϕ is defined by a stochastic integral or stochastic differential equation.

We conclude this note in showing how such an identity can be found in our formulation. For simplicity we choose ϕ to be given as a one dimensional stochastic integral

$$\phi(t) = \int_0^t e(s)dB(s),$$

($e(\cdot)$ nonanticipating). In white-noise language the stochastic integral is

$$\varphi(t)=\int_0^t \partial_s^* e(s)ds,$$

[2, 3]. We easily find

$$\partial_u \varphi(t)=\int_0^t \partial_s^*(\partial_u e(s))ds+e(u) \tag{2.6}$$

and hence

$$\langle\langle \varphi(t), \varphi(t)\rangle\rangle=\int_0^t \left\{e(s)^2+2e(s)\int_0^t \partial_u^*(\partial_s e(u))du+\left(\int_0^t \partial_u^*(\partial_s e(u))du\right)^2\right\}ds. \tag{2.7}$$

A little computation yields the result

$$\langle\langle \varphi(t), \varphi(t)\rangle\rangle=\int_0^t \{e(s)^2+\langle\langle e(s), e(s)\rangle\rangle+2\partial_s^*\langle\langle e(s), \varphi(s)\rangle\rangle\}ds. \tag{2.8}$$

It is now rather straightforward to parallel Stroock's treatment [6] of Malliavin's calculus, in particular to apply it to stochastic differential equations.

References

[1] Hida, T.: Brownian Motion. Springer Verlag (1980).

[2] ——: Brownian Functionals and the Rotation Group. Mathematics + Physics (ed. L. Streit), World Scientific Publ. Co. (1985).

[3] Kubo, I. and Takenaka, S.: Calculus on Gaussian White Noise, I–IV. Proc. Japan Acad., **56**A, 376–380; 411–416 (1980); **57**A, 433–437 (1981); **58**A, 186–189 (1982).

[4] Kuo, H.-H.: Brownian Functionals and Applications. Acta Appl. Math., **1**, 175–188 (1983).

[5] Malliavin, P.: Stochastic Calculus of Variation and Hypoelliptic Operators. Stochastic Differential Equations (ed. K. Itô). Kinokuniya, Tokyo and Wiley, New York (1978).

[6] Stroock, D.: The Malliavin Calculus and its Applications. Stochastic Integrals (ed. D. Williams). Springer LNM 851, Springer Verlag (1981).