116. On Surfaces of Class VII_0 with Curves. II

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Introduction. This is a preliminary report on [6]. A compact complex surface is in class VII_0 if it is minimal and if its first Betti number is equal to one. We know many examples of surfaces of class VII_0 (or in short, VII_0 surfaces) with the second Betti number b_2 positive—minimal surfaces with global spherical shells [1]–[3], [7]–[9]. In view of the table [5, (10.3)], it is necessary to study VII_0 surfaces with a cycle of rational curves in detail in order to complete the classification of VII_0 surfaces.

The main consequences to report are as follows. Let S be a VII_0 surface with C a cycle of rational curves. Then $b_2(S)$, the second Betti number of S, is positive, the deformation functor of S is unobstructed and the cycle C is deformed into a nonsingular elliptic curve in a suitable smooth family of deformations of the surface S. An arbitrary deformation of S with a smooth elliptic curve which is a lifting of C is either a blown-up parabolic Inoue surface or generically a blown-up primary Hopf surface (§1, see also [5, (12.3)]). If moreover S has at least b_2 (possibly singular) rational curves, then S has exactly b_2 rational curves and the weighted dual graph of all the curves on S is completely determined. The dual graph turns out to be the same as one of dual graphs of all the curves on minimal surfaces with global spherical shells (§2). As its consequence, Inoue surfaces with b_2 positive are characterized in a uniform manner (§3).

Notations. We use the usual notations in analytic geometry or the same notations as in [5]. In addition to these, we use the following. $[a, b]:=\{k \in \boldsymbol{Z}; a \leqq k \leqq b\}$, $L_I:=\sum_{i \in I} L_i$, $A \sim B$ iff $c_1(A)=c_1(B)$ in $H^2(S, \boldsymbol{Z})$ for $A, B \in H^1(S, \mathcal{O}_S^*)$.

§1. Smoothing a cycle of rational curves by deformations.

(1.1) **Theorem.** *Let S be a* VII_0 *surface with b_2 positive. Then* $H^2(S, \Theta_S)=0$.

Proof. Assume the contrary to derive a contradiction. By Serre duality, $H^0(S, \Omega_S^1(K_S)) \neq 0$. Let D be the maximum effective divisor of S such that $H^0(S, \Omega_S^1(K_S-D)) \neq 0$ and let ω be a nonzero element of $H^0(S, \Omega_S^1(K_S-D))$. By definition, zero (ω) is isolated. Then the following is exact

$$0 \longrightarrow \mathcal{O}_S(K_S-D) \xrightarrow{f} \Omega_S^1 \xrightarrow{g} \mathcal{O}_S(2K_S-D)$$

where $f(a)=a\omega$, $g(b)=b \wedge \omega$. Let $\mathcal{H}$ be Coker g. Then supp$(\mathcal{H})$ is isolated points, so that $H^q(S, \mathcal{H})=0$ for any $q>0$. Therefore by taking Euler-Poincaré characteristics, we see

$$b_2=-\chi(S, \Omega_S^1)=-\chi(S, -K_S+D)-\chi(S, 2K_S-D)+\chi(S, \mathcal{H})$$

$$=-2K_S^2+3K_SD-D^2+h^0(S,\mathcal{H}).$$

Therefore by $-K_S^2=b_2$, we have $b_2+3K_SD-D^2+h^0(S,\mathcal{H})=0$. We also have $K_SD\geqq 0$, $-D^2\geqq 0$, so that $b_2=0$, $K_SD=D^2=h^0(S,\mathcal{H})=0$. This contradicts the assumption $b_2>0$. Q.E.D.

(1.2) **Theorem.** *Let S be a* VII$_0$ *surface with C a cycle of rational curves, E a reduced effective divisor containing C. Then $H^2(S,\Theta_S(-\log E))=0$.*

From [5, (12.3) or (12.5)] and (1.2), we infer

(1.3) **Theorem.** *Let S be a* VII$_0$ *surface with C a cycle of rational curves, $E=C+H$ a reduced divisor containing C. Then there is a smooth proper family $\pi:\mathcal{S}\to\Delta$ with π-flat divisors $\mathcal{C}$ and $\mathcal{H}$ of S such that*

(1.3.1) $(\mathcal{S}_0,\mathcal{C}_0,\mathcal{H}_0)\cong(S,C,H)$,

(1.3.2) *$\mathcal{H}_t=H$ for any $t\in\Delta$,*

(1.3.3) *$\varpi(:=\pi_{|\mathcal{C}}):\mathcal{C}\to\Delta$ is a versal deformation of C.*

From [5, (12.3) or (12.5)] and (1.3), we infer

(1.4) **Theorem.** *Let S be a* VII$_0$ *surface with C a cycle of rational curves. Then there is a smooth proper family $\pi:\mathcal{S}\to\Delta$ over a unit disc Δ with a π-flat Cartier divisor $\mathcal{C}$ such that*

(1.4.1) $(\mathcal{S}_0,\mathcal{C}_0)\cong(S,C)$,

(1.4.2) *$\mathcal{S}_t$ is a blown-up primary Hopf surface with a nonsingular elliptic curve $\mathcal{C}_t$ $(t\neq 0)$.*

(1.5) **Corollary.** *Let S be a* VII$_0$ *surface with C a cycle of rational curves with $C^2<0$. Suppose that S is not a half Inoue surface. Then there are complex line bundles L_j on S $(1\leqq j\leqq n)$ (which we call a canonical basis) such that*

(1.5.1) *$E_j:=c_1(L_j)$ $(1\leqq j\leqq n)$ is a $\boldsymbol{Z}$-basis of $H^2(S,\boldsymbol{Z})$,*

(1.5.2) $K_SL_j=-1$, $L_jL_k=-\delta_{jk}$,

(1.5.3) *$C=-(L_{r+1}+\cdots+L_n)$, $K_S=L_1+\cdots+L_n$ in $H^1(S,\mathcal{O}_S^*)$ for some $1\leqq r\leqq n-1$, where $n=b_2(S)$.*

§ 2. Dual graphs of curves.

(2.1) **Definition.** A VII$_0$ surface S with $b_2>0$ is called *special* if S has at least b_2 rational curves.

By [5, (3.5)], any special VII$_0$ surface has exactly b_2 rational curves. *Any* VII$_0$ *surface with a global spherical shell is special.* See [1], [3], [6], [7].

(2.2) **Lemma.** *An arbitrary special* VII$_0$ *surface has a cycle of rational curves. It has a canonical basis L_j $(1\leqq j\leqq b_2(S))$.*

(2.3) **Definition.** A reduced connected divisor D is called *a branch* of the cycle C if $CD=1$ and if D has no common components with C.

(2.4) **Theorem.** *Let S be a special* VII$_0$ *surface with $C=C_1+\cdots+C_s$ a cycle of s rational curves. Then we have,*

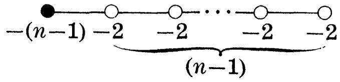
$$(K_S+C)^2=-b_2(C)\ (=-s),\qquad b_2(C)-C^2=b_2(S).$$

(2.5) **Theorem.** *Let S be a special* VII$_0$ *surface with C a rational curve with a node. If C has an irreducible branch D_2, then by indexing the remaining curves D_j $(3\leqq j\leqq n)$ and a canonical basis L_j $(1\leqq j\leqq n)$ sui-*

tably, we have

$$C=-(L_2+\cdots+L_n),\quad D_j\sim L_j-L_{j-1}\quad (2\leqq j\leqq n)$$

where $n=b_2(S)$. The dual graph of n curves is given by



where ● *(resp.* ○*) stands for C (resp. D_j).*

(2.6) **Theorem.** *Let S be a special* VII_0 *surface with $C=C_1+C_2$ a cycle of two rational curves. If S has an irreducible curve D_3 with $C_2D_3=1$, then by indexing suitably, we have one of the following cases:*

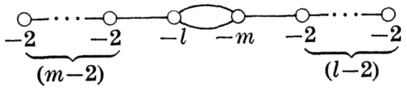
(2.6.1) $C_1\sim L_1-L_2-L_{[3,l]}$, $C_2\sim L_2-L_1-L_{[l+1,l+m-2]}$,
$D_j\sim L_j-L_{j-1}$ $(3\leqq j\leqq l$, *or* $l+2\leqq j\leqq l+m-2)$, $D_{l+1}\sim L_{l+1}-L_1$,

(2.6.2) $C_1\sim L_1-L_2-L_{[3,l]}$, $C_2\sim L_2-L_1-L_{[l+1,l+m-2]}$,
$D_j\sim L_j-L_{j-1}$ $(3\leqq j\leqq l$, *or* $l+2\leqq j\leqq l+m-2)$,
$D_{l+1}\sim L_{l+1}-L_l-L_1$,

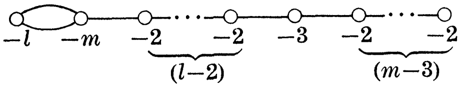
(2.6.3) $C_1\sim L_1-L_2$, $C_2\sim L_2-L_1-L_{[3,n]}$,
$D_3\sim L_3-L_1-L_2$, $D_j\sim L_j-L_{j-1}$ $(4\leqq j\leqq n)$,

(2.6.4) $C_1\sim L_1-L_2-L_{[3,n]}$, $C_2\sim L_2-L_1$, $D_j\sim L_j-L_{j-1}$ $(3\leqq j\leqq n)$

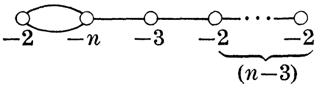
where $l, m, n\geqq 3$ and $b_2(S)$ equals $l+m-2$ or n, C_1D_{l+1} equals 1 (resp. 0) in (2.6.1) (resp. (2.6.2)). The dual graph of b_2 curves is given as follows:

(2.6.5)


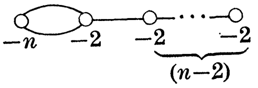
$(l, m\geqq 3)$

(2.6.6)


$(l, m\geqq 3)$

(2.6.7)


$(n\geqq 3)$

(2.6.8)


$(n\geqq 3)$

(2.7) **Theorem.** *Let S be a special* VII_0 *surface with C a cycle of rational curves having branches. Then the dual graph of all the curves on S is the same as one of the dual graphs of curves on minimal surfaces with global spherical shells.*

§ 3. Inoue surfaces with b_2 positive.

(3.1) **Lemma.** *Let S be a* VII_0 *surface with C a unique cycle of rational curves. If $mK_S+D\sim 0$ for an effective divisor D on S, then* supp D *is connected and it contains C.*

(3.2) **Theorem.** *Let S be a special* VII_0 *surface with C a unique cycle of rational curves. Assume $C^2<0$ and that C has no branches. Then S is isomorphic to a half Inoue surface.*

Proof. Since S is special, there are a positive integer m and a divisor D such that $mK_S+D\sim 0$. *By* (3.1) and the assumption, $D_{red}=C$. Hence $D=\sum_i n_iC_i$, C_i being irreducible components of C. Then $(D-mC)C_i=$

$-m(K_S+C)C_i=0$, whence $D=mC$ because (C_iC_j) is negative definite. Hence $b_2(S)=-K_S^2=-C^2$. It follows from [5, (9.3)] that S is isomorphic to a half Inoue surface. Q.E.D.

(3.3) **Definition** [1]. Let S be a VII$_0$ surface with b_2 positive. The *Dloussky invariant* $DI(S)$ of S is

$$DI(S)=-\sum_D D^2+2\sharp\ (\text{rational curves with nodes})$$

D running over all irreducible curves on S.

(3.4) **Theorem.** *Let S be a* VII$_0$ *surface with b_2 positive. Then*

$$DI(S)\leqq 3b_2(S),$$

equality holding iff S is an Inoue surface with b_2 positive.

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