104. On the Derived Categories of Mixed Hodge Modules

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Let X be a nonsingular (separated) algebraic variety over $\boldsymbol{C}$, and $MHM(X, \boldsymbol{Q})$ the abelian category of mixed Hodge Modules [5]. For simplicity, $MHM(X, \boldsymbol{Q})$ will be denoted by $MHM(X)$. Let $D^b MHM(X)$ be the derived category of bounded complexes of $MHM(X)$. Then $D^b MHM(X)$ are stable by the functors: $f_*, f_!, f^*, f^!, \psi_g, \varphi_{g,1}, \xi_g$ (cf. 1.1), $\boldsymbol{D}$ and $\boxtimes$. I would like to thank Prof. Kashiwara for useful and stimulating discussions.

§1. Vanishing cycle functors.

1.1. Let g be a function on X. By definition (cf. [5]) we have the exact functors

$$\psi_g : MHM(X) \longrightarrow MHM(X), \qquad \varphi_{g,1} : MHM(X) \longrightarrow MHM(X).$$

We define a functor $\xi_g : MHM(X) \to MHM(X)$ as follows:

Let $j_g : \{g \neq t\} \to X \times \boldsymbol{C}$ be the open immersion, and $p : X \times \boldsymbol{C} \to X$ the projection, where t is the coordinate of $\boldsymbol{C}$. Then we define

$$\xi_g = \psi_{t,1} j_{g!} j_g^* p^*[1].$$

Note that the functors $j_{g!}$ and $p^*[1]$ exist by definition [5].

1.2. Proposition. *We have the functorial exact sequences:*

$$0 \longrightarrow \psi_{g,1}\mathcal{M} \longrightarrow \xi_g \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow 0$$
$$0 \longrightarrow j_! j^* \mathcal{M} \longrightarrow \xi_g \mathcal{M} \longrightarrow \varphi_{g,1}\mathcal{M} \longrightarrow 0$$

for $\mathcal{M} \in MHM(X)$, *where* $j : X \setminus g^{-1}(0) \to X$.

1.3. Remark. Beilinson's functor Ξ_g used in [1] should correspond to $\xi_g j_*$.

1.4. Corollary. *Let Z be a closed (reduced) subvariety of X, and $MHM_Z(X)$ (resp. $D^b_Z MHM(X)$) the full subcategory of $MHM(X)$ (resp. $D^b MHM(X)$) of the objects with supports (resp. cohomological supports) in Z. Then*

$$D^b MHM_Z(X) \longrightarrow D^b_Z MHM(X)$$

is an equivalence of categories.

This follows from 1.2. by the same argument as in [1], because the adjunction $\mathrm{Hom}(j^*\mathcal{M}^\cdot, \mathcal{N}^\cdot) \simeq \mathrm{Hom}(\mathcal{M}^\cdot, j_*\mathcal{N}^\cdot)$ for an affine open immersion j follows from the existence of the natural morphism $\mathcal{M}^\cdot \to j_* j^* \mathcal{M}^\cdot$.

§2. Duals.

2.1. Proposition. *$MHM(X)$ (hence $D^b MHM(X)$) is stable by the dual functor $\boldsymbol{D}$.*

This follows from the compatibility of the algebraic and topological dualities with respect to the functors ψ, φ_1.

§3. **Direct images.**

3.1. Let $f:X\to Y$ be a morphism of smooth (separated) algebraic varieties. If X is affine, $\mathcal{H}^0 f_*$ (cf. [5]) is right exact and we can derive this by the same argument as in [1], because $f_*(M,F)$ is strict for $(M,F)\in MF(\mathcal{D}_X)$ underlying a mixed Hodge Module, if f is *proper* [4, 5]. In general, we define

$$f_*: D^b MHM(X)\longrightarrow D^b MHM(Y)$$

using an affine Čech covering, cf. [1]. Set $f_!=\boldsymbol{D}f_*\boldsymbol{D}$.

3.2. **Lemma.** *For $\mathcal{M}^\cdot\in D^b MHM(X)$ such that $(\mathcal{H}^i f_*)\mathcal{M}^j=0$ for $i\neq 0$, $f_*\mathcal{M}^\cdot$ is represented by $(\mathcal{H}^0 f_*)\mathcal{M}^\cdot$.*

This follows from the definition (using a result in [1]), because $(\mathcal{H}^i f_*)\mathcal{M}=H^i(f_*\mathcal{M})$ for $\mathcal{M}\in MHM(X)$.

3.3. **Corollary.** *$f_*\simeq f_!$ if f is proper.*

This follows from the compatibility of the algebraic and topological dualities with respect to the proper direct images.

§4. **Pull-backs.**

4.1. Let f be as in §3. We define f^* by the left adjoint functor of f_* and $f^!$ by $\boldsymbol{D}f^*\boldsymbol{D}$, then $f^!$ is the right adjoint of $f_!$. For $g:Y\to Z$, we have $(gf)_*\simeq g_*f_*$, hence $(gf)^*$ exists and is represented by f^*g^*, if f^* and g^* exist. Note that j^* is represented by the usual restriction, if j is an open immersion.

4.2. **Proposition.** *Let $i:X\to Y$ be a closed immersion, and $j:Y\backslash X\to Y$ the open immersion, then i^* exists and we have the canonical triangle:*

$$\longrightarrow j_!j^*\mathcal{M}^\cdot\longrightarrow\mathcal{M}^\cdot\longrightarrow i_*i^*\mathcal{M}^\cdot\xrightarrow{+1}.$$

This follows from 1.4.

4.3. **Proposition.** *Let $p:X\times Y\to Y$ be the projection, then p^* exists and is represented by the functor $\boxtimes a_X^*\boldsymbol{Q}^H$, where $a_X:X\to pt$ and $\boldsymbol{Q}^H\in MHM(pt)$ is the object in* [5, Theorem 1.8].

We can construct $p^*p_*\to id$ and $id\to p_*p^*$, and verify the compatibility condition (the construction of $p^*p_*\to id$ is due to Kashiwara).

4.4. **Remark.** The condition: $\mathrm{Gr}_i^W H^j\mathcal{M}^\cdot=0$ for $i>j$ (resp. $i<j$) is stable by the functors: $f_!, f^*$ (resp. $f_*, f^!$). If $\mathcal{M}$ and $\mathcal{N}$ are pure of weight m and n, $\mathrm{Ext}^i(\mathcal{M},\mathcal{N})=0$ for $m<n+i$.

4.5. **Remark.** We can extend these construction to a singular variety Z in X. In particular we have $a_Z^*\boldsymbol{Q}^H\in D^b MHM(Z)$ $(=D^b MHM_Z(X)$, cf. 1.4) and $a_{Z*}a_Z^*\boldsymbol{Q}^H\in D^b MHM(pt)$. Note that $MHM(pt)$ coincides with the category of graded polarizable $\boldsymbol{Q}$-mixed Hodge structures, cf. [2]. We can also eliminate the condition of embedability, using a covering with local embeddings, cf. [4].

§5. **Extensions.**

5.1. Let X and g be as in §1. Set $Z=g^{-1}(0)$ and $U=X\backslash Z$. Then we have an analogue to Deligne-MacPherson-Verdier's theory on extension of perverse sheaves:

5.2. Proposition. *$MHM(X)$ is equivalent to the category of the objects*: $\{M' \in MHM(U),\ M'' \in MHM_Z(X),\ u \in \mathrm{Hom}(\psi_{g,1} j_* M', M''),\ v \in \mathrm{Hom}(M'', \psi_{g,1} j_* M'(-1)) : vu = N\}$ *where* $j : U \to X$.

This follows from 1.2. (An explicit construction of the inverse functor is obtained by Kashiwara.) We have also MacPherson's version because of the following:

5.3. Lemma. *Let p be as in 4.3, then a mixed Hodge Module on $X \times Y$ is a pull-back of an object on Y, iff the underlying perverse sheaf is.*

5.4. Remark. By 5.2, the proof of the stability by $\boxtimes$ is reduced to the case of local systems. Then the assertion follows from Kashiwara's theory on admissible variation of mixed Hodge structures [3] and the coincidence of the two categories (this coincidence implies the conjecture in the introduction of [4]).

§ 6. Cycle classes.

6.1. Let $Z \subset X$ be a closed irreducible subvariety of dimension d_Z. Put $\boldsymbol{Q}_X^H = a_X^* \boldsymbol{Q}^H \in D^b MHM(X)$ and $\boldsymbol{Q}_Z^H = a_Z^* \boldsymbol{Q}^H \in D^b MHM_Z(X)$. Because $\mathrm{Gr}_i^W H^j \boldsymbol{Q}_Z^H = 0$ for $j > d_Z$ or $i > j$, and $\mathrm{Gr}_{d_Z}^W H^{d_Z} \boldsymbol{Q}_Z^H$ is the intermediate direct image $\underline{IC}_Z \boldsymbol{Q}^H$, we have the morphism: $\boldsymbol{Q}_Z^H \to \underline{IC}_Z \boldsymbol{Q}^H[-d_Z]$. Because we have $\boldsymbol{Q}_X^H \to \boldsymbol{Q}_Z^H$ by adjunction, we get:

$$\boldsymbol{Q}_X^H \longrightarrow \underline{IC}_Z \boldsymbol{Q}^H[-d_Z] \quad \text{and} \quad \underline{IC}_Z \boldsymbol{Q}^H[-d_Z] \longrightarrow \boldsymbol{Q}_X^H(p)[2p]$$

by duality, where p is the codimension. Their composition

$$cl_Z^H \in \mathrm{Hom}(\boldsymbol{Q}_X^H, \boldsymbol{Q}_X^H(p)[2p]) \simeq \mathrm{Hom}(\boldsymbol{Q}^H, a_{X*} \boldsymbol{Q}_X^H(p)[2p])$$

is called the cycle class of Z. This element in the second group coincides with the composition of $\boldsymbol{Q}^H \to a_{Z*} \boldsymbol{Q}_Z^H$ and the direct image by a_{X*} of $\boldsymbol{Q}_Z^H \to \underline{IC}_Z \boldsymbol{Q}^H[-d_Z] \to \boldsymbol{Q}_X^H(p)[2p]$. Let $\pi : Y \to Z$ be a resolution of singularity. Then we have $\boldsymbol{Q}_X^H \to \pi_* \boldsymbol{Q}_Y^H$ and $\pi_* \boldsymbol{Q}_Y^H \to \boldsymbol{Q}_X^H(p)[2p]$ (by duality), and their composition coincides with cl_Z^H in the first group. By Beilinson [6], $\mathrm{Ext}^i(\mathcal{M}, \mathcal{N}) = 0$ for $\mathcal{M}, \mathcal{N} \in MHM(pt)$ if $i > 1$, hence $\mathrm{Hom}(\boldsymbol{Q}^H, a_{X*} \boldsymbol{Q}_X^H(p)[2p])$ is isomorphic to Deligne's cohomology if X smooth proper. (It seems cl_Z^H coincides with the usual cycle map; i.e. it induces the Abel-Jacobi map.) If X is singular, we replace $\boldsymbol{Q}_X^H(p)[2p]$ by $(\boldsymbol{D}\boldsymbol{Q}_X^H)(-d_Z)[-2d_Z]$. Then the $\boldsymbol{Q}$-part $cl_Z^{\boldsymbol{Q}}$ of cl_Z^H belongs to $H^{-2d_Z}(X, \boldsymbol{D}\boldsymbol{Q}_X(-d_Z))$, that is $H^{2p}(X, \boldsymbol{Q}_X(p))$ if X smooth, and $H_{2d_Z}(X, \boldsymbol{Q})(-d_Z)$ if X proper.

References

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