103. *Mixed Hodge Modules*

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Introduction. We define $MHM(X, k)^{(g)}$ the categories of (*geometric*) *mixed Hodge Modules* in the algebraic case, and prove the stability by subquotients, vanishing cycle functors, direct images, pull-backs (and external products). In this note, X, Y are smooth algebraic varieties (assumed always separated) over $\boldsymbol{C}$, and $\mathcal{D}_X$ is the sheaf of algebraic differential operators; all the $\mathcal{D}_X$-Modules are assumed quasi-coherent, and the holonomic Modules regular.

§ 1. Definitions and main results.

1.1. Let $\boldsymbol{k}$ be a subfield of $\boldsymbol{R}$. Let $MF_h(\mathcal{D}_X, \boldsymbol{k})$ be the category of filtered holonomic $\mathcal{D}_X$-Modules (M, F) with $\boldsymbol{k}$-structure given by $DR(M) \simeq \boldsymbol{C} \otimes K$ for $K \in \operatorname{Perv}(\boldsymbol{k}_X)$, $MH_Z(X, \boldsymbol{k}, n)^p$ the category of (algebraic) *polarizable Hodge Modules* of weight n with strict support Z, and $MH(X, \boldsymbol{k}, n)^p := \oplus MH_Z(X, \boldsymbol{k}, n)^p$ (cf. [4, 5]). $MHW(X, \boldsymbol{k})^p$ is the category of the objects of $MF_h(\mathcal{D}_X, \boldsymbol{k})$ with a finite filtration W such that $\mathrm{Gr}_i^W \in MH(X, \boldsymbol{k}, i)^p$ for any i.

1.2. Let g be a function on X. Then by definition

$$\psi_g(M, F, K) = (\oplus_{-1 \le \alpha < 0} (Gr_\alpha^V \tilde{M}, F[1]),\ \psi_g K[-1]),$$
$$\phi_{g,1}(M, F, K) = ((\mathrm{Gr}_0^V \tilde{M}, F),\ \phi_{g,1} K[-1]),$$

for $(M, F, K, W) \in MHW(X, \boldsymbol{k})^p$, where $(\tilde{M}, F) = i_{g*}(M, F)$ with i_g the immersion by graph, and V is the filtration of Malgrange-Kashiwara (cf. [loc. cit]). Let L be the filtration defined by $L_i \psi_g = \psi_g W_{i+1}$ and $L_i \phi_{g,1} = \phi_{g,1} W_i$. We say that the vanishing cycle functors ψ_g and $\phi_{g,1}$ are *well-defined* for $(M, F, K, W) \in MHW(X, \boldsymbol{k})^p$, if the following conditions are satisfied (compare to [6]):

(1.2.1) (F, W, V) are compatible filtrations (cf. [5]) of $\tilde{M}$,

(1.2.2) the monodromy filtration W of ψ_g and $\phi_{g,1}$ relative to L exists (cf. [3]),

(1.2.3) $\operatorname{can}(W_i \psi_{g,1}) \subset W_i \phi_{g,1}$ and $\operatorname{Var}(W_i \phi_{g,1}) \subset W_{i-2} \psi_{g,1}(-1)$,

(1.2.4) (F, W, L) are compatible filtrations of ψ_g and $\phi_{g,1}$,

(1.2.5) $(\psi_g(M, F, K), W)$, $(\phi_{g,1}(M, F, K), W) \in MHW(X, \boldsymbol{k})^p$.

(As is pointed out by Kashiwara, (1.2.3–5) follows from the other conditions.)

1.3. Let $i: U \to X$ be an open immersion such that $X \backslash U$ is a divisor. Let $E = (M, F, K, W) \in MHW(U, \boldsymbol{k})^p$. Then $E' = (M', F, K', W) \in MHW(X, \boldsymbol{k})^p$

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will be denoted by i_*E (resp. $i_!E$), if $K'\simeq i_*K$ (resp. $i_!K$), and if ψ_g and $\phi_{g,1}$ are well-defined for E' for any (local) defining equation of $X\backslash U$.

1.4. Lemma. *i_*E (resp. $i_!E$) is unique if it exists.*

1.5. We define the full subcategories $MHW(X,\boldsymbol{k})^p_{(j)}$ of $MHW(X,\boldsymbol{k})^p$ for $j\geq 0$ by induction on j:

(1.5.1) $E\in MHW(X,\boldsymbol{k})^p_{(0)}$ iff ψ_g and $\phi_{g,1}$ are well-defined for any g (locally defined on X),

(1.5.2) $E\in MHW(X,\boldsymbol{k})^p_{(j)}$ iff, for any open subset U, open immersion $i:U\to U'$ as in 1.3, and function g on U, ψ_g and $\phi_{g,1}$ are well-defined for $E|_U$, and $i^*, i_!$ exist, and belong to $MHW(U',\boldsymbol{k})^p_{(j-1)}$, for $\psi_g E|_U$ and $\phi_{g,1}E|_U$ $(j>0)$.

Set $MHW(X,\boldsymbol{k})^p_{(\infty)}=\bigcap MHW(X,\boldsymbol{k})^p_{(i)}$.

1.6. We define $MHM(X,\boldsymbol{k})$ the categories of mixed Hodge Modules by:

(1.6.1) $E\in MHM(X,\boldsymbol{k})$ iff $(\Omega^d_Y, F, \boldsymbol{k}_Y[d], W)\boxtimes E\in MHW(X\times Y,\boldsymbol{k})^p_{(\infty)}$ for any smooth Y, where $\mathrm{Gr}^F_i\Omega^d_Y=0$ for $i\neq -d$, $\mathrm{Gr}^W_i(\Omega^d_Y,\boldsymbol{k}_Y[d])=0$ for $i\neq d$, and $d=\dim Y$.

1.7. Theorem. *The categories $MHM(X,\boldsymbol{k})$ are abelian categories such that the morphisms are always strict, and stable by subquotients in $MHW(X,\boldsymbol{k})^p$, and by the operations: $\mathcal{H}^jf_*$, $\mathcal{H}^jf_!$, $\mathcal{H}^jf^*$, $\mathcal{H}^jf^!$ for any $j\in\boldsymbol{Z}$ and any morphism f, and $\psi_g,\phi_{g,1}$ for any g (locally defined on X).*

Outline of proof. We verify the first assertion and the stability by subquotients, using [5, §1]. For the pull-backs, we use:

$$i^*=C(\mathrm{can}:\psi_1[-1]\to\phi_1[-1]),\qquad i^!=C(\mathrm{Var}:\phi_1[-2]\to\psi_1(-1)[-2])$$

in the case of closed immersion of codimension one; in general, the well-definedness follows from the stability by quasi-projective direct images. As to the direct images, in the case f proper and E pure, we can reduce to the case f projective, then the general case follows from [5].

1.8. Theorem. $(\boldsymbol{C},F,\boldsymbol{k},W)\in MHM(pt,\boldsymbol{k})$, *where* $\mathrm{Gr}^F_i=\mathrm{Gr}^W_i=0$ *for* $i\neq 0$.

For the proof, we use the result in the next section.

1.9. Let $MHM(X,\boldsymbol{k})^g$ be the smallest full subcategories of $MHM(X,\boldsymbol{k})$ which are stable by the operations in Theorem 1.7, and contain the object in Theorem 1.8. (They correspond to the mixed perverse sheaves of geometric origin [1] in char. p.) Then we have:

1.10. Proposition. *The categories of geometric mixed Hodge Modules $MHM(X,\boldsymbol{k})^g$ are stable also by external products $\boxtimes$.*

§2. Normal crossing case.

2.1. Let $X=\boldsymbol{C}^n$, $D=\{x_1\cdots x_n=0\}$ and $D_I=\{x_i=0\ \ (i\in I)\}$. Let $\mathrm{Perv}(\boldsymbol{C}_X)_{nc}$ be the category of perverse sheaves whose characteristic varieties are contained in the union of conormal bundles of D_I. For $\nu\in(\boldsymbol{C}/\boldsymbol{Z})^n$, we set $\bar{\nu}=\{i\in\bar{n}:\nu_i\neq 0\}$, where $\bar{n}:=\{1,\cdots,n\}$. For $\nu\in(\boldsymbol{C}/\boldsymbol{Z})^n$, $I\subset\bar{n}\backslash\bar{\nu}$, and $\mathcal{F}\in\mathrm{Perv}(\boldsymbol{C}_X)_{nc}$, we define:

$$\mathcal{F}^\nu_I=\Psi^{\nu_1}_{x_1}\cdots\Psi^{\nu_n}_{x_n}\mathcal{F},\qquad\text{where } \Psi^{\nu_i}_{x_i}=\psi^{\nu_i}_{x_i}[-1](i\notin I),\ \phi^0_{x_i}[-1](i\in I).$$

Here ψ^{α} (resp. ϕ^{α}) means $\mathrm{Ker}\,(T_s-\exp(2\pi i\alpha))$ with T_s the semi-simple part of the monodromy T. Set $N=\log T_u/(2\pi i)$, and for each i, let can_i, Var_i and N_i be the morphisms associated to the functors ψ_{x_i} and ϕ_{x_i}. For $g=x^m$, set $\overline{m}=\{i : m_i\neq 0\}$. Then:

2.2. Proposition. *We have the canonical isomorphisms as* $\boldsymbol{C}[N]$-*modules*:

$$(\psi_g^{\alpha}\mathcal{F})_I^{\nu}\simeq \mathrm{Coker}\,[(N_*-m_*N)_{I\cap m} : \mathcal{F}_{I\setminus m}^{\nu+\alpha m}[N]\to\mathcal{F}_{I\setminus m}^{\nu+\alpha m}[N]]$$

$$(\phi_g^{0}\mathcal{F})_I^{\nu}\simeq \mathrm{Coker}\left[\begin{pmatrix}((N_*-m_*N)_{I\cap m}-N_{I\cap m})N^{-1}, & -\mathrm{Var}_{I\setminus m}^{I} \\ \mathrm{can}_I^{I\setminus m} \quad , & N\end{pmatrix} : \begin{pmatrix}\mathcal{F}_{I\setminus m}^{\nu}[N] \\ \oplus\mathcal{F}_I^{\nu}[N]\end{pmatrix}\to\begin{pmatrix}\mathcal{F}_{I\setminus m}^{\nu}[N] \\ \oplus\mathcal{F}_I^{\nu}[N]\end{pmatrix}\right]$$

where $(N_*-m_*N)_J=\prod_{i\in J}(N_i-m_iN)$, $N_J=\prod_{i\in J}N_i$, $\mathrm{Var}_{I\setminus m}^{I}=\prod_{i\in I\cap m}\mathrm{Var}_i$ *and* $\mathrm{can}_I^{I\setminus m}=\prod_{i\in I\cap m}\mathrm{can}_i$. (*We can also express explicitly* can_i, Var_i, N_i, *and* can, Var *for* $\psi\mathcal{F}$ *and* $\phi\mathcal{F}$ *in terms of those morphisms for* $\mathcal{F}$.)

For the proof, we use $\mathcal{D}$-Modules; we can also treat the Hodge filtration and prove the stability of some condition by vanishing cycle functors (at least) in the following case:

2.3. Corollary. *Let W be a finite filtration of* $\mathcal{F}\in\mathrm{Perv}\,(\boldsymbol{C}_X)_{nc}$ *such that* Gr_i^W *are semi-simple and that* $N_i(W_j\mathcal{F})_I^{\nu}\subset(W_{j-2}\mathcal{F})_I^{\nu}$ *for any i, j, then the relative monodromy filtration W of* $\psi_g\mathcal{F}$ *and* $\phi_{g,1}\mathcal{F}$ *exists, where* $g=x^m$, *and* $(\psi_g\mathcal{F}, W)$, $(\phi_{g,1}\mathcal{F}, W)$ *satisfy the same condition.* (*Moreover, the condition* (1.2.3) *is also stable, and the filtration L* (cf. 1.2) *on* Gr^W *splits* (*globally*).)

§ 3. Remarks.

3.1. In the analytic case, one can not expect the stability by the direct images for Zariski open immersions. Except for this, we have a similar theory: in (1.5.2) $U=U'$, and in Theorem 1.7, f is projective for direct images. We can also consider the case X are quasi-projective over a fixed complex manifold S. This includes the case studied by many people: e.g. El Zein, Steenbrink-Zucker, Guillen-Navarro-Puerta, Du Bois, etc., where S is a disc and $f : X\to S$.

3.2. For a closed subvariety Z of X, the uniqueness of the Hodge filtration of the filtered De Rham complex over $\boldsymbol{C}_Z$ (due to Du Bois) is a direct consequence of the strictness of the Hodge filtration on the corresponding complex of $\mathcal{D}$-Modules: the same proof as in [2] applies. We can also show that Dec W (for $\mathcal{D}$-Modules!) is well-defined.

3.3. Let Z be a projective variety with an ample invertible sheaf L, imbedded in $X=\boldsymbol{P}^N$ by a power of L, then, for a mixed Hodge Module (M, F, K, W) on X, whose support is contained in Z, we have the "Kodaira vanishing":

$$H^i(Z, \mathrm{Gr}^F(DR_X(M, F))\otimes L^{\pm 1})=0 \qquad \text{for } i\gtrless 0.$$

This gives a generalization of a result of Kollar (for $\mathrm{Gr}^F=R^jf_*\omega_Y$ with Y smooth projective) and of Guillen-Navarro-Puerta (for the filtered De Rham complex over $\boldsymbol{C}_Z$, because $\boldsymbol{C}_Z[\dim Z]$ is semi-perverse).

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