94. Representations of a Solvable Lie Group on $\bar{\partial}_b$ Cohomology Spaces

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Let $(\mathfrak{g}, j, \omega)$ be a normal j-algebra introduced by Pyatetskii-Shapiro [5] (see below). We denote by G the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. The aim of this note is to give, relating its construction to a certain geometric structure, a unitary representation of G in which every irreducible (up to a set of Plancherel measure zero) occurs with multiplicity one.

1. A triplet $(\mathfrak{g}, j, \omega)$ of a completely solvable Lie algebra $\mathfrak{g}$, a linear operator j on $\mathfrak{g}$ such that $j^2=-1_{\mathfrak{g}}$ and $\omega \in \mathfrak{g}^*$ is termed a *normal j-algebra* if (i) the Nijenhuis tensor for j vanishes, (ii) $\langle x, y\rangle := \omega([x, jy])$ defines an inner product on $\mathfrak{g}$ relative to which j is an orthogonal transformation. Let $G=\exp \mathfrak{g}$, the connected and simply connected Lie group corresponding to $\mathfrak{g}$. As is well-known, there is a Siegel domain D of type II on which G acts simply transitively by affine automorphisms. Denote by $S(D)$ the Šilov boundary of D. Then, $S(D)$ is diffeomorphic to a nilpotent (at most 2-step) normal subgroup $N(D)$ of G. Moreover, G is written as a semi-direct product $G=N(D) \rtimes G(0)$ with a closed subgroup $G(0)$ of G. We assume throughout this note that D does not reduce to a tube domain. In this case, $N(D)$ is a 2-step nilpotent Lie group and $S(D)$ has a natural CR structure. So, the tangential Cauchy-Riemann operator $\bar{\partial}_b$ is defined and we have $\bar{\partial}_b \circ \bar{\partial}_b=0$.

By Rossi-Vergne [7], the unitary representations of $N(D)$ defined by translations on the square integrable $\bar{\partial}_b$ cohomology spaces H^q $(q=0,1,\cdots)$ on $S(D)$ contain almost every irreducible of $N(D)$. We will define unitary representations of G on H^q $(q=0,1,\cdots)$ such that their restrictions to $N(D)$ coincide with those of [7]. We remark that there is no G-invariant Riemannian metric on $S(D)$, so the usual geometric method is not directly applicable.

2. It is known that $\mathfrak{g}$ is written as an orthogonal direct sum (relative to $\langle\cdot,\cdot\rangle$) $\mathfrak{g}=\mathfrak{g}(0)\oplus\mathfrak{g}(1/2)\oplus\mathfrak{g}(1)$ with $[\mathfrak{g}(k), \mathfrak{g}(m)]\subset\mathfrak{g}(k+m)$, where we understand $\mathfrak{g}(k)=\{0\}$ for $k>1$. Then, $\mathfrak{g}(0)=\mathrm{Lie}\, G(0)$ and we have $\mathfrak{n}(D):=\mathrm{Lie}\, N(D)$ $=\mathfrak{g}(1/2)+\mathfrak{g}(1)$. We put $V=\mathfrak{g}(1/2)$. Then V is j-invariant, so $\dim V>0$ is even. We denote by Ξ the set of all $\lambda \in \mathfrak{g}(1)^*$ such that the skew-symmetric bilinear form $\lambda([x, y])$ on V is non-degenerate. Ξ is an open dense subset of $\mathfrak{g}(1)^*$. Let J be a Borel mapping with values in real linear operators on V such that for each $\lambda \in \Xi$, (i) $J(\lambda)$ is a complex structure on V satisfying

$\lambda([J(\lambda)x, J(\lambda)y])=\lambda([x, y])$ for all $x, y \in V$, (ii) the quadratic form $\lambda([J(\lambda)v, v])$ on V is negative definite. Through the holomorphic induction, we obtain a measurable (with respect to the Lebesgue measure on $\mathfrak{g}(1)^*$) family $(U_{\lambda,J(\lambda)}, \mathfrak{F}_{\lambda,J(\lambda)})_{\lambda\in\Xi}$ of irreducible unitary representations (IURs) of $N(D)$. Here $\mathfrak{F}_{\lambda,J(\lambda)}$ is the Hilbert space of entire functions F on the complex vector space $(V, J(\lambda))$ such that

$$\int_V |F(v)|^2 \exp \frac{1}{2}\lambda([J(\lambda)v, v])d\mu(v)<\infty,$$

where $d\mu(v)$ is the Lebesgue measure on V.

For every $\lambda \in \mathfrak{g}(1)^*$, let A_λ be the skew-symmetric linear operator on V defined by $\lambda([x, y])=4\langle A_\lambda x, y\rangle$ $(x, y \in V)$. We set $P(\lambda)=(\det A_\lambda)^{1/2}$. Evidently, $P(\lambda)>0$ for $\lambda \in \Xi$. We have the following decomposition of $L^2(N(D))$, by which the double regular representation of $N(D)$ is decomposed.

Theorem 1. *With suitable normalizations of the relevant measures, there is a unitary mapping Φ from $L^2(N(D))$ onto $\int_\Xi^\oplus \boldsymbol{B}_2(\mathfrak{F}_{\lambda,J(\lambda)})P(\lambda)d\lambda$ such that for $f \in L^1(N(D)) \cap L^2(N(D))$*

$$\Phi f(\lambda)=\int_{N(D)} f(n)U_{\lambda,J(\lambda)}(n)^{-1}dn,$$

where $\boldsymbol{B}_2(\mathfrak{F}_{\lambda,J(\lambda)})$ is the Hilbert space of the Hilbert-Schmidt operators on $\mathfrak{F}_{\lambda,J(\lambda)}$.

3. Set $V^\pm=V_C(j\,;\pm i)$. Then, $V^\pm$ turn out to be abelian subalgebras of $\mathfrak{n}(D)_C$. Hence V^+ defines a left invariant CR structure on $N(D)$. The CR manifold $N(D)$ thus obtained is CR isomorphic to $S(D)$. Let $\langle\cdot,\cdot\rangle_C$ denote the hermitian inner product on $\mathfrak{n}(D)_C$ obtained by extending $\langle\cdot,\cdot\rangle$. Then, $V^\pm$ are mutually orthogonal relative to $\langle\cdot,\cdot\rangle_C$, so that we have an orthogonal decomposition $\mathfrak{n}(D)_C=\mathfrak{g}(1)_C\oplus V^+\oplus V^-$. Hence is defined the $\bar{\partial}_b$ Laplacian $\Box_b$ acting on $C_c^\infty(N(D))\otimes\wedge^q V^+$ $(q=0, 1, \cdots)$.

4. To analyze $\Box_b$, we pick a specific family of IURs of $N(D)$. Put $(x, y)=\omega([x, jy])-i\omega([x, y])$ $(x, y \in V)$. Then, $(\cdot,\cdot)$ defines a hermitian inner product on the complex vector space $(V, j|_V)$. For $\lambda \in \mathfrak{g}(1)^*$, let H_λ be the selfadjoint operator on $(V, j|_V)$ associated with the hermitian quadratic form $\lambda([jz, z])/4$ $(z \in (V, j|_V))$. Let $|H_\lambda|=(H_\lambda^2)^{1/2}$. It is clear that if $\lambda \in \Xi$, H_λ is non-singular. We now define a family of complex linear operators j_λ $(\lambda \in \Xi)$ on $(V, j|_V)$ by $j_\lambda=-i|H_\lambda|^{-1}H_\lambda$. Regarding j_λ as real linear operators on V, we have a Borel mapping $\lambda\to j_\lambda$ which satisfies (i), (ii) in 2. Therefore we get a measurable family $(U_\lambda, \mathfrak{F}_\lambda)_{\lambda\in\Xi}$ of IURs of $N(D)$.

Let $\Box_b^q$ be the closure in $L^2(N(D))\otimes\wedge^q V^+$ of the operator $\Box_b$ on $C_c^\infty(N(D))\otimes\wedge^q V^+$. The closed subspace $H^q:=\operatorname{Ker}\Box_b^q$ is called the *q-th square integrable $\bar{\partial}_b$ cohomology space.* By Theorem 1, we have

$$L^2(N(D))\simeq\int_\Xi^\oplus \boldsymbol{B}_2(\mathfrak{F}_\lambda)P(\lambda)d\lambda$$

and by [1, Proposition 11, p. 174] this isomorphism extends to the isomorphism

$$\Phi_q : L^2(N(D))\otimes\wedge^q V^+ \rightrightarrows \int_{\Xi}^{\oplus} \boldsymbol{B}_2(\mathfrak{F}_\lambda)\otimes V^{+,q}(\lambda)P(\lambda)d\lambda,$$

where $V^{+,q}(\lambda)$ is the constant field of Hilbert spaces over Ξ defined by $V^{+,q}(\lambda)=\wedge^q V^+$. We call the unitary mapping Φ_q the *Fourier transformation.*

For each $q=0,1,\cdots$, let Ξ_q be the set of all $\lambda\in\Xi$ such that the self-adjoint operator H_λ has q negative eigenvalues. Ξ_q is an open (possibly empty) subset of $\mathfrak{g}(1)^*$. We denote by A_λ the closed subspace of all $T\in\boldsymbol{B}_2(\mathfrak{F}_\lambda)$ such that Range $T\subset\mathbf{C}1$, where $1\in\mathfrak{F}_\lambda$ is the constant function with value 1. On the other hand, noting that j_λ leaves V^+ invariant, we let $V^+(j_\lambda; i)$ be the i-eigenspace of j_λ in V^+ and put $\delta(\lambda)=\dim_C V^+(j_\lambda; i)$. Set $\mathfrak{Z}(\lambda)=\wedge^{\delta(\lambda)}V^+(j_\lambda; i)$. Then, if $\lambda\in\Xi_q$, we have $\delta(\lambda)=q$, so that $\mathfrak{Z}(\lambda)\subset\wedge^q V^+$. It is clear that $\lambda\to\mathfrak{Z}(\lambda)$ is a measurable field of one dimensional Hilbert spaces.

Theorem 2. *The Fourier transformation Φ_q induces a unitary mapping from H^q onto*

$$\mathcal{H}^q := \int_{\Xi_q}^{\oplus} A_\lambda\otimes\mathfrak{Z}(\lambda)P(\lambda)d\lambda.$$

Corollary (cf. [7]). *$H^q=\{0\}$ if and only if $\Xi_q=\varnothing$.*

5. Now it is easy to see that $A_\lambda\simeq\mathfrak{F}_{-\lambda}$ canonically as Hilbert spaces, so that we have

$$\mathcal{H}^q \simeq \boldsymbol{H}^q := \int_{\Xi_{n-q}}^{\oplus} \mathfrak{F}_\lambda P(\lambda)d\lambda,$$

where $2n=\dim V$. We will define a unitary representation of G on $\boldsymbol{H}^q$ (hence on H^q). We note here that $G(0)$ acts on $\mathfrak{g}(1)^*$ with 2^l open orbits O_η ($\eta\in\mathfrak{X}:=\{-1,1\}^l$), where l is the rank of the normal j-algebra $(\mathfrak{g}, j, \omega)$ (cf. [6, Proposition 3.3.1]). For each $\eta\in\mathfrak{X}$ we can construct a continuous mapping $J(\cdot,\eta)$ defined on $G(0)$ such that for any $(g,\eta)\in G(0)\times\mathfrak{X}$, $J(g,\eta)$ satisfies, in addition to (i) and (ii) in **2** with $\lambda=g\cdot\lambda_\eta$ ($\lambda_\eta\in O_\eta$ chosen suitably), the following relation:

(1) $\quad (\mathrm{Ad}_V\, g_1)\circ J(g_2,\eta)=J(g_1g_2,\eta)\circ(\mathrm{Ad}_V\, g_1) \qquad$ for all $g_1, g_2\in G(0)$.

Thus we get another family $(U_{g,\eta}, \mathfrak{F}_{g,\eta})$ ($g\in G(0), \eta\in\mathfrak{X}$) of IURs of $N(D)$, which is measurable with respect to the left Haar measure on $G(0)$ for every $\eta\in\mathfrak{X}$.

Since $U_{g,\eta}$ is unitarily equivalent to U_λ ($\lambda=g\cdot\lambda_\eta$) defined in **4**, there is a unitary intertwining operator $\mathcal{J}(\lambda):\mathfrak{F}_\lambda\to\mathfrak{F}_{g,\eta}$. $\mathcal{J}(\lambda)$ is given explicitly by an integral operator (cf. [3]). On the other hand, let

$$\mathcal{R}(g_0)F(v)=[\det \mathrm{Ad}_V\, g_0]^{-1/2}F(g_0^{-1}\cdot v) \qquad (g_0\in G(0)).$$

Then, owing to (1), $\mathcal{R}(g_0)$ is a unitary mapping from $\mathfrak{F}_{g,\eta}$ onto $\mathfrak{F}_{g_0 g,\eta}$ for arbitrary $g_0, g\in G(0)$ and $\eta\in\mathfrak{X}$. We put

$$\mathcal{R}_0(g_0;\lambda)=\mathcal{J}(g_0\cdot\lambda)^{-1}\mathcal{R}(g_0)\mathcal{J}(\lambda).$$

Then, it is easy to see that $\mathcal{R}_0(g;\lambda)$ is a unitary mapping from $\mathfrak{F}_\lambda$ onto $\mathfrak{F}_{g\cdot\lambda}$ and satisfies $\mathcal{R}_0(g_1g_2;\lambda)=\mathcal{R}_0(g_1; g_2\cdot\lambda)\mathcal{R}_0(g_2;\lambda)$. Since we have an IUR U_λ of $N(D)$ on $\mathfrak{F}_\lambda$, we can thus define unitary representations τ_q of $G=N(D)\rtimes S(0)$ on $\boldsymbol{H}^q$ (hence on H^q) ($q=0,1,\cdots$).

Theorem 3. *Let* σ_G *be the Kirillov-Bernat mapping* $\mathfrak{g}^*/G \to \hat{G}$. *Then, denoting by* $[\tau_q]$ *the equivalence class of* τ_q, *one has* $[\tau_q] = \sum^{\oplus}_{\eta \in \mathfrak{X}_{n-q}} \sigma_G(G \cdot \lambda_\eta)$, *where* $\mathfrak{X}_{n-q} = \{\eta \in \mathfrak{X}\,;\, O_\eta \subset \Xi_{n-q}\}$.

Remark. τ_0 is the quasi-regular representation of G on the square integrable CR functions on $N(D)$.

It can be shown that $\{G \cdot \lambda_\eta\,;\, \eta \in \mathfrak{X}\}$ exhausts all open coadjoint orbits in $\mathfrak{g}^*$. Combining Theorem 3 with [2, p. 132], we get

Theorem 4. $\sum^{\oplus}_{0 \leq q \leq n} \tau_q$ *contains all* (*except for a set of Plancherel measure zero*) *irreducible unitary representations of* G *exactly once.*

Finally we remark that the IUR belonging to each $\sigma_G(G \cdot \lambda_\eta)$ is square integrable by [2, Théorème 5.3.4].

The details of this note will appear elsewhere.

References

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