79. The n-Plane with Integral Curvature Zero and without Conjugate Points

By Nobuhiro INNAMI
Faculty of Integrated Arts and Sciences, Hiroshima University

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0. Introduction. Cohn-Vossen ([2]) proved that a plane without conjugate points has the nonpositive integral Gaussian curvature if it exists ([1]). Recently, Green-Gulliver ([4]) has proved that a plane whose metric differs from the canonical flat metric at most on a compact set is Euclidean if there is no conjugate point. The assumption implies automatically that its integral Gaussian curvature is zero. The purpose of the present note is to show the following

Theorem. *Let M be an n-plane without conjugate points. If the absolute integral Ricci curvature exists and the integral scalar curvature is zero, then M is Euclidean.*

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1. Preliminaries. Let M be a complete simply connected Riemannian manifold without conjugate points. Then M is diffeomorphic to the Euclidean space E^n, $n=\dim M$, and all geodesics are minimizing.

The limit solution of matrix Jacobi equation. Let $\gamma:(-\infty,\infty)\to M$ be a geodesic. Let R be the curvature tensor of M. We consider the $(n-1)\times(n-1)$ matrix Jacobi equation

$$D''(t)+R(t)D(t)=0, \tag{$*$}$$

where $R(t):\dot{\gamma}(t)^{\perp}\to\dot{\gamma}(t)^{\perp}$ is given by $R(t)(x)=R(x,\dot{\gamma}(t))\dot{\gamma}(t)$ for any $x\in\dot{\gamma}(t)^{\perp}$. It follows from [3] that if $D_s(t)$, $s>0$, is the solution of $(*)$ with $D_s(0)=I$, $D_s(s)=0$, then $D(t)=\lim_{s\to\infty}D_s(t)$ exists for any $t\in(-\infty,\infty)$ and is a solution of $(*)$ such that $D(0)=I$, $\det(D(t))\neq 0$ for any $t\in(-\infty,\infty)$. Set $A(t)=D'(t)D^{-1}(t)$ for any $t\in(-\infty,\infty)$. Then it follows that $A(t)$ is symmetric and

$$A'(t)+A(t)^2+R(t)=0$$

for any $t\in(-\infty,\infty)$.

The volume form of the unit tangent bundles. Let SM be the unit tangent bundle. Let $f^t:SM\to SM$ be the geodesic flow, i.e., $f^tv=\dot{\gamma}_v(t)$, where $\gamma_v:(-\infty,\infty)\to M$ is the geodesic with $\dot{\gamma}(0)=v\in SM$. It is well-known that $d\omega=d\sigma\wedge d\theta$ is f^t-invariant, where $d\sigma$ is the volume form induced from the Riemannian metric and $d\theta$ is canonical volume form on the unit sphere in E^n, $n=\dim M$. We introduce an equivalence relation $\sim$ in such a way that $v\sim w$ if $v=f^tw$ for some $t\in(-\infty,\infty)$, where $v,w\in SM$. Let N be the set of all equivalence classes $[v]$, $v\in SM$. All orbits $[v]$, $v\in SM$, are images of minimizing geodesics in SM because of minimality of all geodesics in M. Hence, for any $v \in SM$ there exists locally a hypersurface H in SM containing v and diffeomorphic to an open subset in E^{2n-2} such that $[w] \cap H = \{w\}$ for any $w \in H$. The collection of such hypersurfaces yields a differentiable structure of N with dimension $2n-2$. We define the volume form $d\eta$ on N such that $d\eta_{[v]} \wedge dt = d\omega_v$ for any $[v] \in N$. Then we have, for any integrable function F on SM,

$$\int_{SM} F\, d\omega = \int_{[v]\in N} d\eta \int_{-\infty}^{\infty} F_{[v]}(f^t v)dt,$$

where $F_{[v]} : [v] \to \boldsymbol{R}$ is given by $F_{[v]}(w) = F(w)$ for any $w \in [v]$.

2. Proof of Theorem. From the existence of the absolute integral Ricci curvature, the absolute Ricci curvature is integrable along the geodesics $\gamma_v : (-\infty, \infty) \to M$ with $\dot{\gamma}_v(0) = v$ for almost all $v \in SM$. Let v be such a vector. We write $A(v)$ and $R(v)$ instead of $A(0)$ and $R(0)$ constructed in Preliminaries, respectively. Then

$$(**) \qquad A'(f^t v) + A(f^t v)^2 + R(f^t v) = 0,$$

for any $t \in (-\infty, \infty)$.

We first prove that there exist sequences $a \to \infty$, $b \to -\infty$ such that $\operatorname{tr} A(f^a v) \to 0$, $\operatorname{tr} A(f^b v) \to 0$ as $a \to \infty$, $b \to -\infty$: Suppose for indirect proof that an $\varepsilon > 0$ and an s exist such that $|\operatorname{tr} A(f^t v)| > \varepsilon$ for any $t > s$. Then

$$\operatorname{tr} A(f^t v) - \operatorname{tr} A(f^s v) + \int_s^t \operatorname{tr} A(f^t v)^2 dt + \int_s^t \operatorname{Ric}(f^t v) dt = 0$$

for any $t > s$, where Ric means the Ricci curvature. Since

$$(\operatorname{tr} A(f^t v))^2 \leq n^2 \operatorname{tr} A(f^t v)^2,$$

we have

$$\int_s^t \operatorname{tr} A(f^t v)^2 dt \geq (\varepsilon/n)^2 (t-s)$$

for any $t > s$, and, hence, $\operatorname{tr} A(f^t v) \to -\infty$ as $t \to \infty$. If we take a $u > s$ such that $|\operatorname{tr} A(f^t v)| > 1$ for any $t \geq u$, then

$$\begin{aligned} \frac{t-u}{n^2} &\leq \int_u^t \frac{\operatorname{tr} A(f^t v)^2}{(\operatorname{tr} A(f^t v))^2} dt \leq \left| \int_u^t \frac{\operatorname{tr} A'(f^t v)}{(\operatorname{tr} A(f^t v))^2} dt \right| \\ &\quad + \left| \int_u^t \frac{\operatorname{Ric}(f^t v)}{(\operatorname{tr} A(f^t v))^2} dt \right| \\ &\leq \left| -\frac{1}{\operatorname{tr} A(f^t v)} + \frac{1}{\operatorname{tr} A(f^u v)} \right| + \int_u^t |\operatorname{Ric}(f^t v)|\, dt, \end{aligned}$$

a contradiction, because the right hand side is bounded above. The existence of a sequence $b \to -\infty$ we want is proved similarly.

Integrating $(**)$ on $[b, a]$ and taking $a \to \infty$, $b \to -\infty$, we obtain

$$\int_{-\infty}^{\infty} \operatorname{tr} A(f^t v)^2 dt = -\int_{-\infty}^{\infty} \operatorname{Ric}(f^t v) dt.$$

By integrating on N,

$$\int_{SM} \operatorname{tr} A(v)^2 d\omega = -\int_{SM} \operatorname{Ric}(v) d\omega = -\frac{\theta_{n-1}}{n} \int_M \operatorname{Sc}\, d\sigma = 0,$$

where θ_{n-1} is the volume of the unit sphere S^{n-1} and Sc is the scalar curvature of M. Therefore, we have

$$\operatorname{tr} A(v)^2=0 \to A(v)=0 \to A'(v)=0 \to R(v)=0 \to R(\cdot, v)v=0$$

for almost all $v \in SM$. Since $R(\cdot, v)v$ is continuous on SM, we conclude that M is flat.

References

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