77. On Conjugacy Classes of the Pro-l braid Group of Degree 2[†)]

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(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1986)

0. Introduction. In [2], Y. Ihara studied the "pro-l braid group" of degree 2 which is a certain big subgroup $\Phi \subset \text{Out}\,\mathfrak{F}$ of the outer automorphism group of the free pro-l group $\mathfrak{F}$ of rank 2. There is a canonical representation $\varphi_Q : G_Q \to \Phi$ of the absolute Galois group $G_Q = \text{Gal}\,(\bar{Q}/Q)$ which is unramified outside l, and for each prime $p \neq l$, the Frobenius of p determines a conjugacy class C_p of Φ which is contained in the subset $\Phi_p \subset \Phi$ formed of all elements of "norm" p (loc. cit. Ch. I). In this note, we shall prove that Φ_p contains *infinitely* many Φ-conjugacy classes, at least if p generates $Z_l^\times$ topologically. It is an open question whether one can *distinguish* the Frobenius conjugacy class from other norm-p-conjugacy classes.

1. The result. Let l be a rational prime. We denote by Z_l, $Z_l^\times$ and Q_l, respectively, the ring of l-adic integers, the group of l-adic units and the field of l-adic numbers. As in [2], let $\mathfrak{F} = \mathfrak{F}^{(2)}$ be the free pro-l group of rank 2 generated by x, y, z, $xyz = 1$, $\Phi = \text{Brd}^{(2)}\,(\mathfrak{F}; x, y, z)$ be the pro-l braid group of degree 2, $\text{Nr}\,(\sigma) \in Z_l^\times$ be the norm of $\sigma \in \Phi$, and for $\alpha \in Z_l^\times$, Φ_α be the "norm-α-part", i.e., $\Phi_\alpha = \{\sigma \in \Phi \mid \text{Nr}\,(\sigma) = \alpha\}$.

Theorem. *If $\alpha \in Z_l^\times$ generates $Z_l^\times$, then the set Φ_α contains infinitely many Φ-conjugacy classes.*

Remarks. 1) In [2], it is proved under the same assumption, that Φ_α contains at least two Φ-conjugacy classes. (Corollary of Proposition 8, Ch. I.)

2) In [1], M. Asada and the author studied the "pro-l mapping class group" and obtained a result similar to 1).

2. Proof. Our method of proof is to consider the projection of Φ to the group $\Psi = \text{Brd}^{(2)}\,(\mathfrak{F}/\mathfrak{F}''; x, y, z)$, where $\mathfrak{F}'' = [\mathfrak{F}', \mathfrak{F}']$, $\mathfrak{F}' = [\mathfrak{F}, \mathfrak{F}]$ and we use the same symbols x, y, z for their classes mod $\mathfrak{F}''$. By Theorem 3 in [2] Ch. II, the group Ψ is explicitly realized as follows. Define the group Θ by

$$\Theta = \{(\alpha, F) \mid \alpha \in Z_l^\times, F \in \mathcal{A}^\times, F + uvw\mathcal{A} = \theta_\alpha\}$$

with the composition law $(\alpha, F)(\beta, G) = (\alpha\beta, F \cdot G^{j\alpha})$, where

$$\mathcal{A} = Z_l[[u, v, w]]/((1+u)(1+v)(1+w)-1) \simeq Z_l[[u, v]],$$

θ_α is certain class mod uvw determined by α, and j_α is a unique automorphism of the Z_l-algebra $\mathcal{A}$ determined by

$$(1+u)\longrightarrow(1+u)^\alpha, \quad (1+v)\longrightarrow(1+v)^\alpha, \quad (1+w)\longrightarrow(1+w)^\alpha.$$

Then, $\Psi\simeq\Theta$ and $\Psi_1\simeq1+uvw\mathcal{A}$. Here, for $\alpha\in Z_l^\times$, Ψ_α is the norm-α-part. Henceforth, we identify Ψ(resp. Ψ_1) with Θ(resp. $1+uvw\mathcal{A}$) by this isomorphism.

Now, we shall prove that if α generates $Z_l^\times$, Ψ_α contains infinitely many Ψ-conjugacy classes.

We fix an element $(\alpha, F_\alpha)\in\Psi_\alpha$. For any $(\alpha, H)\in\Psi_\alpha$, write

$$H=F_\alpha(1+uvwH_0), \qquad H_0\in\mathcal{A}.$$

Since α generates $Z_l^\times$, the centralizer of (α, H) in Ψ contains an element with arbitrary norm. Thus, in Ψ_α, Ψ-conjugacy is equivalent to Ψ_1-conjugacy. Let

$$G=1+uvwG_0\in\Psi_1, \qquad G_0\in\mathcal{A}.$$

Then

(1) $$G^{-1}(\alpha, H)G=(\alpha, HG^{j_\alpha}G^{-1})\in\Psi_\alpha$$

and

(2) $$HG^{j_\alpha}G^{-1}=F_\alpha(1+uvwH_0)(1+uvwG_0)^{j_\alpha}(1+uvwG_0)^{-1}.$$

If we write

(3) $$HG^{j_\alpha}G^{-1}=F_\alpha(1+uvwJ), \qquad J\in\mathcal{A},$$

we get

(4) $$J\equiv H_0+(uvw)^{j_\alpha-1}G_0^{j_\alpha}-G_0 \qquad \text{mod } uvw.$$

Now, identify $\mathcal{A}$ with $Z_l[[u, v]]$ and write

$$G_0 \text{ mod } u=b_0+b_1v+b_2v^2+\cdots, \qquad b_i\in Z_l \ (i\geq0).$$

We view b_i $(i\geq0)$ as variables over Z_l. Direct calculation shows that we can write

(5) $$(uvw)^{j_\alpha-1}G_0^{j_\alpha}-G_0 \text{ mod } u=\sum_{i=0}^\infty\{(\alpha^{i+3}-1)b_i+Q_i(b_0, b_1, \cdots, b_{i-1})\}v^i$$

where Q_i is a linear form determined alone by α with coefficients in Z_l in i variables. (Put $Q_0=0$.) For (α, H), $(\alpha, H')\in\Psi_\alpha$, write

$$H=F_\alpha(1+uvwH_0), \quad H'=F_\alpha(1+uvwH_0'), \quad H_0, H_0'\in\mathcal{A},$$
$$H_0 \text{ mod } u=h_0+h_1v+h_2v^2+\cdots, \quad H_0' \text{ mod } u=h_0'+h_1'v+h_2'v^2+\cdots, \quad h_i, h_i'\in Z_l,$$
$$h(H)=(h_0, h_1, h_2, \cdots), \qquad h(H')=(h_0', h_1', h_2', \cdots).$$

Then by (1)–(5), if (α, H) and (α, H') are Ψ_1-conjugate to each other, there exist $b_i\in Z_l$, $i=0, 1, 2, \cdots$, such that

(6) $$h_i=h_i'+(\alpha^{i+3}-1)b_i+Q_i(b_0, b_1, \cdots, b_{i-1}) \qquad \text{for all } i.$$

In view of this, we shall define an equivalence relation in $Z_l^\infty=\{h=(h_0, h_1, h_2, \cdots)\,|\,{}^\forall h_i\in Z_l\}$. For $h=(h_0, h_1, h_2, \cdots)\in Z_l^\infty$ and $i\geq3$, define an element $R_i(h)\in \boldsymbol{Q}_l$ inductively by

(7) $$R_i(h)=\frac{1}{\alpha^i-1}\{h_{i-3}-Q_{i-3}(R_3(h), R_4(h), \cdots, R_{i-1}(h))\}.$$

It follows from (6) that, for $h=(h_0, h_1, \cdots)$, $h'=(h_0', h_1', \cdots)\in Z_l^\infty$ corresponding to H_0, H_0',

(8) $$b_i=R_{i+3}(h)-R_{i+3}(h') \qquad (i\geq0).$$

(Note that Q_i is a linear form.) Since α generates $Z_l^\times$, $\alpha^i-1\in Z_l^\times$ unless

$l-1\,|\,i$. So, for any integer $k \geq 1$, define

$$h \overset{(k)}{\sim} h' \text{ if and only if } R_{i(l-1)}(h) - R_{i(l-1)}(h') \in \boldsymbol{Z}_l \text{ for any } i$$
$$\text{satisfying } 1 \leq i \leq k.$$

This is an equivalence relation in $\boldsymbol{Z}_l^\infty$. We call its equivalence class (k)-equivalence class. Therefore $(\alpha, H) \sim (\alpha, H')$ ($\mathit{\Psi}_1$-conjugate to each other) implies $h(H) \overset{(k)}{\sim} h(H')$ for all $k \geq 1$.

We shall show that the number of (k)-equivalence classes in $\boldsymbol{Z}_l^\infty$ tends to infinity as $k \to \infty$. Let $k \geq 2$ and $l^\nu \| k$, i.e., l^ν is the exact power of l dividing k. Then $(\alpha^{k(l-1)}-1)\boldsymbol{Z}_l = l^{\nu+1}\boldsymbol{Z}_l$. We claim that a $(k-1)$-equivalence class consists of $l^{\nu+1}$ distinct (k)-equivalence classes. To see this, we fix a manner of "l-adic expansion" of an element in $\boldsymbol{Q}_l$, i.e., for $a \in \boldsymbol{Q}_l$, we write $a = \sum_{i=-m}^\infty a_i l^i \in \boldsymbol{Q}_l$, $a_i \in \boldsymbol{Z}$, $0 \leq a_i \leq l-1$, $m \in \boldsymbol{Z}$. We define the "fractional part" $\{a\}$ of a as $\sum_{i=-m}^{-1} a_i l^i$. Then $h \overset{(k)}{\sim} h'$ is equivalent to

$$\{R_{i(l-1)}(h)\} = \{R_{i(l-1)}(h')\} \quad \text{for all } i, \quad 1 \leq i \leq k.$$

Put

$$\tilde{R}_i(h) = \{R_{i(l-1)}(h)\}.$$

If h runs through a $(k-1)$-equivalence class, $Q_{k(l-1)-3}(0, \cdots, 0, \tilde{R}_1(h), 0, \cdots, 0, \tilde{R}_2(h), 0, \cdots, 0, \tilde{R}_{k-1}(h), 0, \cdots, 0)$ is independent of h and the sum of this element and $(\alpha^{k(l-1)}-1)R_{k(l-1)}(h)$ belongs to $\boldsymbol{Z}_l$. By the definition of $R_{k(l-1)}(h)$, we see easily that this sum takes every value mod $l^{\nu+1}$ ($l^\nu \| k$) as h varying in a $(k-1)$-equivalence class. Therefore, a $(k-1)$-equivalence class consists of $l^{\nu+1}$ distinct (k)-equivalence classes and hence the number of (k)-equivalence class in $\boldsymbol{Z}_l^\infty$ tends to infinity as $k \to \infty$. By definition, the map $\mathit{\Psi}_\alpha \ni (\alpha, H) \to h(H) \in \boldsymbol{Z}_l^\infty$ is surjective. Therefore, we have shown that, if $\alpha \in \boldsymbol{Z}_l^\times$ generates $\boldsymbol{Z}_l^\times$, the set $\mathit{\Psi}_\alpha$ contains infinitely many $\mathit{\Psi}$-conjugacy classes.

Next, we shall deduce the theorem from this. Let

$$\mathit{\Psi}^- = \{(\alpha, F) \in \Theta \,|\, F\bar{F} = \alpha(uvw)^{j\alpha-1}\}, \quad \mathit{\Psi}_\alpha^- = \mathit{\Psi}^- \cap \mathit{\Psi}_\alpha \quad (\alpha \in \boldsymbol{Z}_l^\times),$$

where $\bar{F} = F^{j-1}$ for $F \in \mathcal{A}$. Let $\gamma : \Phi \to \mathit{\Psi}$ be the natural map induced from $\operatorname{Aut}\mathfrak{F} \to \operatorname{Aut}(\mathfrak{F}/\mathfrak{F}'')$. Then, by Theorem 8 in [2] Ch. IV, the image of γ coincides with $\mathit{\Psi}^-$. So, it suffices to show that there are infinitely many elements in $\mathit{\Psi}_\alpha^-$ which are not $\mathit{\Psi}_1$-conjugate to each other. We may choose our (α, F_α) from the minus part $\mathit{\Psi}_\alpha^-$ of $\mathit{\Psi}_\alpha$. Let $(\alpha, H) \in \mathit{\Psi}_\alpha^-$ and write $H = F_\alpha(1+uvwH_0)$, $H_0 \in \mathcal{A}$. Then $1+uvwH_0 \in \mathit{\Psi}_1^-$. It follows from this that $H_0 \equiv \bar{H}_0 \bmod u$. Conversely, for $H_0 \in \mathcal{A}$ satisfying $H_0 \equiv \bar{H}_0 \bmod u$, there exists $1+uvwH_0' \in \mathit{\Psi}_1^-$ such that $H_0' \equiv H_0 \bmod u$. This can be seen in the same way as in the proof of Proposition 1 (ii), Ch. III, [2]. Therefore, when H runs through $\mathit{\Psi}_\alpha^-$, i.e., $1+uvwH_0$ runs through $\mathit{\Psi}_1^-$, $H_0 \bmod u$ runs through every element satisfying $H_0 \equiv \bar{H}_0 \bmod u$. Now let

$$H_0 \bmod u = h_0 + h_1 v + h_2 v^2 + \cdots.$$

The condition $H_0 \equiv \bar{H}_0 \bmod u$ is satisfied if and only if h_{2i}, $i = 0, 1, 2, \cdots$, are arbitrary and h_{2i+1}, $i = 0, 1, 2, \cdots$, are determined inductively by the relations

(9) $h_1=0, \quad h_{2i+1}+{}_iC_1\cdot h_{2i}+{}_iC_2\cdot h_{2i-1}+\cdots+{}_iC_{i-1}\cdot h_{i+2}+h_{i+1}=0 \quad (i\geq 1).$

This can be seen easily by expanding

$$\bar{H}_0 \bmod u = h_0 - h_1 v(1-v+v^2-\cdots)+h_2 v^2(1-v+v^2-\cdots)^2-\cdots$$

and comparing the coefficient of v^i for $i=0,1,2,\cdots$. So, to prove the theorem, it suffices to show that when $h_0, h_2, h_4, \cdots$, vary freely in Z_l and $h_1, h_3, h_5, \cdots$, are determined by (9), the number of (k)-equivalence classes to which h belongs tends to infinity as $k\to\infty$. As before, this can be checked by a lengthy but straightforward calculation of the quantity

$$(\alpha^{k(l-1)}-1)R_{k(l-1)}(h)$$
$$+Q_{k(l-1)-3}(0, \cdots, 0, \tilde{R}_1(h), 0, \cdots, 0, \tilde{R}_2(h), 0, \cdots, 0,$$
$$\tilde{R}_{k-1}(h), 0, \cdots, 0) \bmod l^{\nu+1}.$$

References

[1]   M. Asada and M. Kaneko: On the automorphism group of some pro-l fundamental group (to appear in Advanced Studies in Pure Math.).

[2]   Y. Ihara: Profinite braid groups, Galois representations and complex multiplications. Ann. of Math., **123**, 43–106 (1986).