73. *On Matukuma's Equation and Related Topics*

By Wei-Ming NI*) and Shoji YOTSUTANI**)

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1986)

§1. **Introduction.** In 1930, Matukuma, an astrophysicist, proposed the following mathematical model to describe the dynamics of a globular cluster of stars,

(M) $$\Delta u+(1+|x|^2)^{-1}u^p=0, \qquad x\in \boldsymbol{R}^3,$$

where $p>1$, u represents the gravitational potential (therefore $u>0$), $\rho=-(4\pi)^{-1}\Delta u=\{4\pi(1+|x|^2)\}^{-1}u^p$ represents the density and $\iiint \rho dx$ represents the total mass (for details, see [1]). Since the globular cluster has the radial symmetry, positive radial entire solutions of (M) (i.e. solutions of (M) with $u(x)=u(|x|)>0$ for all $x\in \boldsymbol{R}^3$) are of particular interest, and the equation (M) reduces to an ordinary differential equation

(M_α) $$u_{rr}+(2/r)u_r+(1+r^2)^{-1}u^p=0 \ (r>0), \quad u(0)=\alpha, \quad u_r(0)=0,$$

where $\alpha>0$. For each $\alpha>0$, we denote the global unique solution of (M_α) by $u=u(r;\alpha)$. Studying the structure of solutions of (M_α), Matukuma conjectured:

(i) if $p<3$, then $u(r;\alpha)$ has a finite zero for every $\alpha>0$,

(ii) if $p=3$, then $u(r;\alpha)$ is a positive entire solution with finite total mass for every $\alpha>0$,

(iii) if $p>3$, then $u(r;\alpha)$ is a positive entire solution with infinite total mass for every $\alpha>0$.

In 1938, Matukuma found an interesting exact solution ([2])

(S) $$u(r;\sqrt{3})=\{3/(1+r^2)\}^{1/2} \qquad (p=3),$$

which confirms part of his conjecture.

It turns out that the equation (M_α) is more delicate than Matukuma had expected. In answer to his conjecture, we prove that

(i) if $1<p<5$, then $u(r;\alpha)$ has a finite zero for every sufficiently large $\alpha>0$,

(ii) if $1<p<5$, then $u(r;\alpha)$ is a positive entire solution with infinite total mass for every sufficiently small $\alpha>0$,

(iii) if $p\geq 5$, then $u(r;\alpha)$ is a positive entire solution with infinite total mass for every $\alpha>0$.

The conclusions above follow from our more general results stated in Section 2 below. (Set $K(r)=1/(1+r^2)$, $n=3$, and $\sigma=0$ in Theorem 4, $l=-2$ and $C=1$ in Theorem 3, and $\sigma=0$ in Theorem 5.) It is rather interesting

*) School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, U.S.A.

**) Department of Applied Science, Faculty of Engineering, Miyazaki University, Miyazaki 880.

to note that not only the exponent p but also the initial value α has vital influence on the behavior of the solution $u(r;\alpha)$.

§2. **Main results.** Consider the initial value problem

$$(\mathrm{P}_\alpha) \qquad u_{rr}+((n-1)/r)u_r+K(r)(u^+)^p=0 \quad (r>0), \quad u(0)=\alpha>0,$$

where $n\geq 3$, and $u^+=\max\{u,0\}$. (Note that we have replaced the nonlinear term u^p in (M_α) by $(u^+)^p$, since we are interested in positive solutions only.) For simplicity, $K(r)$ will always be assumed to be nonnegative and be continuously differentiable in $(0,\infty)$. Some additional hypotheses on $K(r)$ are collected in the following which will be assumed under various circumstances (but not simultaneously) in this section.

(H1) $\quad K(r)=O(r^\sigma) \quad$ at $r=0$,

(H2) $\quad \liminf_{r\to\infty}\{r^{-l}K(r)\}>0, \quad \lim_{r\to\infty}\{r(r^{-l}K(r))_r\}=0$,

(H3) $\quad K(r)=Cr^l+o(r^l) \quad$ at $r=\infty$, $\quad \lim_{r\to\infty}\{r(r^{-l}K(r))_r\}=0$,

(H4) $\quad K(r)=Cr^\sigma+o(r^\sigma) \quad$ at $r=0$,

(H5) $\quad (r^\sigma K(r))_r\leq 0 \quad$ on $\boldsymbol{R}_+$,

where σ, l and C are constants satisfying

$$\sigma>-2, \quad l\geq -2, \quad C>0.$$

Remark. Suppose that (H1) holds. Then (P_α) possesses a unique solution $u(r;\alpha)\in C([0,\infty))\cap C^2((0,\infty))$. Moreover, we have

(i) if $\sigma>-1$, then $u\in C^1([0,\infty))\cap C^2((0,\infty))$ and $u_r(0)=0$.

(ii) if $K\in C([0,\infty))$, then $u\in C^2([0,\infty))$ and $u_r(0)=0$.

Hereafter we denote the global unique solution of (P_α) by $u(r;\alpha)$. For the existence of positive entire solutions, we have the following two theorems.

Theorem 1. *Suppose that* (H1) *holds with* $\sigma=0$, $K(r)=O(r^{-2})$ *at* $r=\infty$ *and* $p>1$. *Then there exists* α_0 *such that for every* $\alpha\in(0,\alpha_0]$, $u(r;\alpha)$ *is positive on* $\boldsymbol{R}_+$ *and* $\lim_{r\to\infty}u(r;\alpha)=0$.

Theorem 2. *Suppose that* (H1), (H2) *and* $p>(n+2+2l)/(n-2)$ *hold. Then there exists* α_0 *such that for every* $\alpha\in(0,\alpha_0]$, $u(r;\alpha)$ *is positive on* $\boldsymbol{R}_+$ *and* $\lim_{r\to\infty}u(r;\alpha)=0$.

Concerning the (corresponding) total mass of the solutions that guaranteed by Theorem 2, we have

Theorem 3. *Suppose that* (H1), (H3) *and* $p>(n+2+2l)/(n-2)$ *hold. Then there exists* $\alpha_0>0$ *such that for every* $\alpha\in(0,\alpha_0]$, $u(r;\alpha)$ *is positive on* $\boldsymbol{R}_+$, $\lim_{r\to\infty}u(r;\alpha)=0$, *and* $\int K(r)u(r)^p r^{n-1}dr=\infty$.

For the behavior of solutions of (P_α) with large initial data or with large exponent, we have

Theorem 4. *Suppose that* (H1), (H4) *and* $p<(n+2+2\sigma)/(n-2)$ *hold. Then there exists* $\alpha_1>0$ *such that for every* $\alpha\geq\alpha_1$, $u(r;\alpha)$ *has a finite zero on* $\boldsymbol{R}_+$.

Theorem 5. *Suppose that* (H1), (H5) *and* $p\geq(n+2+2\sigma)/(n-2)$ *hold. Then for every* $\alpha>0$, $u(r;\alpha)$ *is positive on* $\boldsymbol{R}_+$. *Moreover if* $(r^{-\sigma}K(r))_r\not\equiv 0$ *on* $\boldsymbol{R}_+$ *or* $p>(n+2+2\sigma)/(n-2)$, *then* $\int K(r)u(r)^p r^{n-1}dr=\infty$.

The equation (P_α) has been studied extensively in recent years, see, for instance, [3]–[10]. The case K decays faster than $O(r^{-2})$ at $r=\infty$ is solved in [3], and is improved slightly by [8]–[9] later. All the methods developed in previous papers do not seem to apply to the case where K decays slower than or equal to $O(r^{-2})$ at $r=\infty$ and $p<(n+2)/(n-2)$. Our results cover most of the cases left open by previous works. We should also remark that all the exponents appeared in Theorems 1–5 are sharp. This may be seen by constructing suitable examples.

All the proofs of the results stated in this section as well as their extensions and other related results will appear in [11].

§3. Concluding remarks. We have given partial answers to the Matukuma's conjecture in §1. Concerning the structure of the solutions of (M_α) with $1<p<5$, our analysis is still incomplete and we suspect that the following conjecture holds: for every $1<p<5$, there exists a unique $\alpha_p>0$ such that

(i) if $\alpha>\alpha_p$, then $u(r;\alpha)$ has a finite zero,

(ii) if $\alpha=\alpha_p$, then $u(r;\alpha)$ is a positive entire solution with finite total mass,

(iii) if $0<\alpha<\alpha_p$, then $u(r;\alpha)$ is a positive entire solution with infinite total mass.

In other words, we conjecture that for $1<p<5$ there is a unique positive entire solution of (M_α) with finite total mass.

Finally we remark that the equation (P_α), with $K(r)\equiv 1$, is known as the Lane-Emden equation and u corresponds to the density of a single star (see [12]). The structure of this equation is simpler than that of Matukuma's equation due to the homogeneity of the nonlinearity. At the critical exponent $p=(n+2)/(n-2)$, (P_α) arises in the problem of finding conformal Riemannian metrics with prescribed scalar curvatures. In this context, the exact solution (S) has also been known for quite some time which represents the usual metric on the standard unit sphere.

Acknowledgements. The second author wishes to thank Professor T. Makino for bringing the equation (M) to his attention. The major part of our work was completed while the second author was visiting the University of Minnesota during 1985–1986 supported by the Ministry of Education, Science and Culture of Japan (60-KOU-337). He would like to thank the staff of the School of Mathematics of University of Minnesota for their hospitality. The first author is partially supported by NSF Grants DMS 8200033A01 and DMS 8601246.

References

[1] T. Matukuma: Dynamics of globular clusters. Nippon Temmongakkai Yoho, 1, 68–89 (1930) (in Japanese).

[2] ——: The Cosmos. Iwanami Shoten (1938) (in Japanese).

[3] W.-M. Ni: On the elliptic equation $\Delta u+K(x)u^{(n+2)/(n-2)}=0$, its generalization, and applications in geometry. Indiana Univ. Math. J., **31**, 493–529 (1982).

[4] ——: Uniqueness, nonuniqueness and related questions of nonlinear elliptic and parabolic equations. Proc. Symp. Pure Math., **44**, 1986 (to appear).

[5] C. E. Kenig and W.-M. Ni: An exterior Dirichlet problem with applications to some nonlinear equations arising in geometry. Amer. J. Math., **106**, 689–702 (1984).

[6] W.-M. Ni and R. D. Nussbaum: Uniqueness and nonuniqueness for positive radial solutions of $\Delta u+f(u,r)=0$. Comm. Pure Appl. Math., **38**, 67–108 (1985).

[7] W.-Y. Ding and W.-M. Ni: On the elliptic equation $\Delta u+Ku^{(n+2)/(n-2)}=0$ and related topics. Duke Math. J., **52**, 485–506 (1985).

[8] N. Kawano: On bounded entire solutions of semilinear elliptic equations. Hiroshima Math. J., **14**, 125–158 (1984).

[9] M. Naito: A note on bounded positive entire solution of semilinear elliptic equations. ibid., **14**, 211–214 (1984).

[10] N. Kawano, J. Satsuma and S. Yotsutani: On the positive solutions of an Emden-type elliptic equation. Proc. Japan Acad., **61A**, 186–189 (1985).

[11] W.-M. Ni and S. Yotsutani: (in preparation).

[12] S. Chandrasekhar: An Introduction to the Study of Steller Structure. University of Chicago Press (1939).