72. *Local Isometric Embedding Problem of Riemannian 3-manifold into* $\mathbf{R}^6$

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§1. **Introduction.** Although the problem of the existence of a local C^∞ isometric embedding for a Riemannian n-manifold (M, g) into Euclidean space $\boldsymbol{R}^{n(n+1)/2}$ is an old and famous problem, there are only a few results if $n \geqq 3$. Recently, Bryant-Griffiths-Yang [1] made a big contribution to the case $n=3$. In this paper, we generalize their results as follows:

Theorem 1. *Let (M, g) be a C^∞ Riemannian 3-manifold and $p_0 \in M$ be a point such that the curvature tensor $\boldsymbol{R}(p_0)$ does not vanish. Then there exists a local C^∞ isometric embedding of a neighborhood U_0 of p_0 into $\boldsymbol{R}^6$.*

The result of [1] treats under the additional assumption:

$(*)$ $\qquad R(p_0)$ does not have signature $(0, 1)$,

where the signature of $R(p)$ is defined by considering $R(p)$ as a symmetric linear operator acting on the space of 2-forms.

§2. **Linearized PDE for the isometric embedding equation.** We shall consider the linearized PDE corresponding to the isometric embedding equation. Take $p_0 \in M$ as the origin and let $U(u^1, u^2, u^3)$ be a coordinate neighborhood around p_0. Let $(x^A(u))$ be a local C^∞ embedding of U into $\boldsymbol{R}^6$ and consider the following PDE for the unknown functions $(y^A(u))$:

(1) $\qquad \nabla_i y_j + \nabla_j y_i = 2\sum_{\lambda=4}^{6} y_\lambda H_{ij\lambda}(u) + k_{ij}(u) \qquad i, j = 1, 2, 3,$

where $(k_{ij}(u))$ is a symmetric 3×3 matrix depending smoothly on u. Here, choosing a unit normal frame field $\{N_\lambda(u)\}_{\lambda=4,5,6}$ on U, we set

$$y^A(u) = \sum_{i=1}^{6} y_i \frac{\partial x^A}{\partial u^i} + \sum_{\lambda=4}^{6} y_\lambda \cdot N_\lambda^A,$$

and denote by ∇ and $H_{ij\lambda}(u)$ the covariant derivatives and the second fundamental form in terms of the isometric embedding $(x^A)_{A=1,\dots,6}$ and the unit normal frame $\{N_\lambda\}$, respectively.

Definition 2. An isometric embedding is called *non-degenerate* if the corresponding second fundamental form $(H_{ij\lambda}(u))$ is linearly independent in the space of all 3×3 symmetric matrices at each point of U.

For a positive integer N, let P be an $N\times N$ system of classical pseudo-differential operator on M with the principal symbol $p(x, \xi)$.

Definition 3. P is called *a system of (real) principal type* at $x_0 \in M$ if, for any $(x_0, \xi_0) \in T^*M - \{0\}$, there exists a conic neighborhood Γ of (x_0, ξ_0), an $N\times N$ homogeneous classical symbol $\tilde{p}(x, \xi)$, and a (real valued) homo-

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geneous classical symbol $q(x, \xi)$ such that

(2) $\tilde{p}(x, \xi)p(x, \xi)=q(x, \xi)Id_N$ in Γ, and dq, ξdx are linearly independent on $\Gamma \cap \{(x, \xi)\,;\, q(x, \xi)=0\}$,

where Id_N is the $N\times N$ identity matrix.

The following is a basic fact for constructing a local isometric em bedding:

Proposition 3. *Under the assumption of Theorem* 1, *given any* $\eta>0$ *and any positive integer s, there exist an open neighborhood* U_0 *of* p_0, *a triplet of non-degenerate* 3×3 *symmetric matrices* $(H_{ij\lambda}(0))$, *a triplet of local* C^∞ *unit vector fields* $(N_\lambda(u))_{\lambda=4,5,6}$ *on* U_0, *and non-degenerate* C^∞ *embedding* $(x^A(u))_{A=1,\dots,6}$ *of* U_0 *into* $\boldsymbol{R}^6$ *such that*

(i) $(N_\lambda(u))_{\lambda=4,5,6}$ *is a unit normal frame field on* U_0 *of the embedding* $(x^A)_{A=1,\cdots 6}$,

(ii) $(H_{ij\lambda}(0))$ *are the second fundamental form of* (x^A) *at* p_0 *with respect to* $(N_\lambda(u))$,

(iii) (1) *is a* 3×3 *real principal type on* U_0 *and*

(iv) $\left\| g_{ij}(u) - \sum_{A=1}^{n(n+1)} \frac{\partial x^A}{\partial u^i}\cdot\frac{\partial x^A}{\partial u^j}\right\|_{H^s(U)} < \eta.$

§3. Local solvability of a non-linear PED of (2). From Proposition 3, one can easily see y_λ ($\lambda=4, 5, 6$) in (1) are determined by an algebraic manipulation if once y_i ($i=1, 2, 3$) are known. Hence, y_λ ($\lambda=4, 5, 6$) play an important role in solving the equation of isometric embedding. In this point of view, (1) essentially reduces to 3×3-system of first order non-linear PDE

$$\Phi(u)=g \tag{3}$$

with $\mathcal{B}^\infty(\boldsymbol{R}^3)$ ($\mathcal{B}^\infty(\boldsymbol{R}^3)$ being the set of C^∞ functions on $\boldsymbol{R}^3$ with bounded derivatives) coefficients whose linearization at u_0 is a system of real principal type. As for the solvability of (3) we have the following:

Theorem 4. *Let* $\Phi(u)$ *be an* $N\times N$ *system of non-linear partial differential operator of order m defined on* $\boldsymbol{R}^n$ *with* $\mathcal{B}^\infty(\boldsymbol{R}^n)$ *coefficient. Assume* $\Phi(u)$ *is Fréchet differentiable in any Sobolev space of order* $\geqq 0$ *and denote its derivative by* $\Phi'(u)$. *Let* $x_0 \in \boldsymbol{R}^n$ *and* $u_0 \in C^\infty(U_0, \boldsymbol{R}^N)$. *Assume that* $\Phi'(u_0)$ *is an* $N\times N$ *system of real principal type at* x_0. *Then, there exists a neighborhood* $U_1 \subset U_0$ *of* x_0, $s_0 \in \boldsymbol{Z}_+$, *and* $\eta>0$ *such that the following property holds: For any* $g \in C^\infty(U_1)$, *satisfying*

$$\|g-\Phi(u_0)\|_{H^{s_0}(U_1)}<\eta, \tag{4}$$

there exists $u \in C^\infty(\boldsymbol{R}^n, \boldsymbol{R}^N)$ *such that* $\Phi(u)=g$ *in* U_1.

§4. The outline of the proof of Theorem 4. As usual our proof is based on the Nash-Moser type implicit function theorem and is proceeded as follows. By checking the argument of Duistermaat-Hörmander [2], we construct an exact local right inverse $Q(u)$ of $\Phi'(u)$ with the properties

$$\|Q(u)h\|_{s-d} \leqq C_s(\|h\|_s + \|h\|_d \|u\|_s), \tag{5}$$

$$\Phi'(u)Q(u)h=h \quad \text{in } U_1, \tag{6}$$

for any real $s\geqq d$ and any $h \in H^s(\boldsymbol{R}^n)$, which are valid if $\|u-u_0\|_\alpha \leqq \delta$ for some large fixed $\alpha>0$ and sufficiently small δ. Here the constants C_s, d and the open neighborhood U_1 of x_0 are independent of u, and $\|\cdot\|_s$ denotes the norm of the Sobolev space $H^s(\boldsymbol{R}^n)$. Next, rewrite

(7) $\Phi(u)=g$ in U_1 in the form:

(8) $\tilde{\Phi}(u)=\tilde{g}$ in U_1

where $\tilde{\Phi}(u)=\Phi(u+u_0)-\Phi(u_0)$, $\tilde{g}=g-\Phi(u_0)$. To solve (8), define a series of functions $\{u_n\}$ by $u_1=0$, $u_{n+1}=u_n+s_{\theta_n}\rho_n$ $(n\geqq 1)$, where $\{s_\theta\}_{\theta\geqq 1}$ are the smoothing operators for the Banach scale $\{H^s(\boldsymbol{R}^n)\}$, $\theta_n=\theta^{\tau^{n+n_0}}$ with $\tau=4/3$ and $\theta>1$, n_0 taken sufficiently large, $\rho_n=Q(u_n+u_0)\Xi\Lambda g_n$, $g_n=\tilde{g}-\tilde{\Phi}(u_n)$, $\Xi: H^s(U_1)\to H^s(\boldsymbol{R}^n)$ is the extension operator and $\Lambda: H^s(\boldsymbol{R}^n)\to H^s(U_1)$ is the restriction operator. Then, assuming $\|\tilde{g}\|_{H^{s_0}(U_1)}$ is sufficiently small for some large s_0, we can prove the following estimates by induction on j:

(i)$_j$ $\|u_j\|_\alpha\leqq\delta$

(ii)$_j$ $\|g_j\|_{H^a(U_1)}\leqq M\theta_j^{-\mu}\|g_1\|_{H^{s_0}(U_1)}$

(iii)$_j$ $\|\rho_j\|_\alpha\leqq M\theta_j^{-a}\|g_1\|_{H^{s_0}(U_1)}$,

for some constants M, μ and a independent of j. With these estimates, we can see the limit $u=\lim_{j\to\infty}u_j$ exists in $H^\alpha(\boldsymbol{R}^n)$ and satisfies (8). Moreover, by the usual interpolation argument, we can prove $u\in C^\infty(\boldsymbol{R}^n)$. Details and proofs will be published elsewhere.

Remark. We note that the recent result on the local isometric embedding problem for the case $n=2$ due to Lin [3] also follows from Theorem 4.

References

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