71. *Estimation of Multiple Laplace Transforms of Convex Functions with an Application to Analytic* (C_0)*-semigroups*

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1. This note is concerned with a new method of estimating multiple Laplace transforms of convex functions of the form

$$(1)\qquad \int_0^\infty \cdots \int_0^\infty \exp(-\textstyle\sum_{i=1}^n \lambda_i \xi_i) f(\sum_{i=1}^n \xi_i) d\xi_1 \cdots d\xi_n,$$

where $\lambda_i > 0$ for $i = 1, \cdots, n$ and $f(\xi)$ is a nonnegative convex function on $(0, \infty)$.

This problem arose from estimating the iteration of resolvents of the infinitesimal generator A of an analytic (C_0)-semigroup $\mathcal{T} = \{T(t) : t \geqq 0\}$ on a Banach space X. Consider the operators

$$(2)\qquad A_\theta \textstyle\prod_{i=1}^n (I - h_i A)^{-1}$$

for $h_i > 0$, $i = 1, \cdots, n$ and $n = 1, 2, \cdots$, where we assume that $\|T(t)\| \leqq Me^{-\omega t}$ for $t \geqq 0$ and some $M \geqq 1$ and $\omega > 0$; $\theta \in (0, 1)$; $A_\theta = -(-A)^\theta$; and $(-A)^\theta$ is the fractional power of $-A$. By means of the relation

$$(I - h_i A)^{-1} x = h^{-1} \int_0^\infty e^{-(\xi/h)} T(\xi) x d\xi, \qquad x \in X,$$

$A_\theta \prod_{i=1}^n (I - h_i A)^{-1} x$ is written as

$$(\textstyle\prod_{i=1}^n h_i^{-1}) \int_0^\infty \cdots \int_0^\infty \exp(-\sum_{i=1}^n h_i^{-1} \xi_i) A_\theta T(\sum_{i=1}^n \xi_i) x d\xi_1 \cdots d\xi_n.$$

Since $\|A_\theta T(\xi)\|$ is dominated pointwise by the convex function $f(\xi) \equiv c_\theta \xi^{-\theta}$ on $(0, \infty)$, c_θ being a positive constant depending only upon θ, the norm of the operator (2) is bounded above by the following type of multiple integral:

$$(3)\qquad (\textstyle\prod_{i=1}^n h_i^{-1}) \int_0^\infty \cdots \int_0^\infty \exp(-\sum_{i=1}^n h_i^{-1} \xi_i) f(\sum_{i=1}^n \xi_i) d\xi_1 \cdots d\xi_n.$$

Our objective here is to describe a new method for estimating the above multiple integrals and show that they are bounded by the value of the integral

$$(4)\qquad (m-1)!^{-1} h^{-m} \int_0^\infty \xi^{m-1} e^{-(\xi/h)} f(\xi) d\xi,$$

provided that $n \geqq m$, $h = m^{-1} \sum_{i=1}^n h_i$ and $h_i \leqq h$ for $i = 1, \cdots, n$.

Let m be any positive integer. Let $m - 1 \leqq \alpha < m$ and consider the function $f(\xi) = c_\alpha \xi^{-\alpha}$ on $(0, \infty)$, where c_α is a positive constant. Then $\int_0^\infty \xi^{m-1} e^{-(\xi/h)} f(\xi) d\xi < \infty$ and the integral (4) with this singular convex function is evaluated as $(m-1)!^{-1} c_\alpha \Gamma(m - \alpha) h^\alpha$, where $\Gamma(s)$ denotes the gamma function. It turns out that given analytic semigroup $\mathcal{T}=\{T(t)\}$ as mentioned above there is C_α such that

$$\|A^{m-1}A_\theta \prod_{i=1}^{n}(I-h_iA)^{-1}\| \leqq C_\alpha(\sum_{i=1}^{n} h_i)^{-\alpha} \tag{5}$$

for $\theta=\alpha-(m-1)$ and h_j, $j=1,\cdots,n$, with $0<h_j\leqq m^{-1}\sum_{i=1}^{n} h_i$. In case $h_1=\cdots=h_n>0$, it is not difficult to derive the estimate (5). See [1] and [2]. However estimates of the form (5) have not been known yet and the application of the estimate (5) yields a new characterization of the infinitesimal generator of an analytic semigroup which involves the characterization due to Crandall, Pazy and Tartar [1, Theorem 1]. See Theorem 3 below. Moreover, it should be mentioned that the estimation (5) is particularly applied to relatively continuous perturbations of analytic semigroups.

2. Let f be a nonnegative convex function on $(0,\infty)$ and consider the multiple integral (3). Since f is continuous on $(0,\infty)$, the integrals under consideration can be taken in the sense of Lebesgue. In what follows, we fix any $t>0$. Let $h_i>0$, $i=1,\cdots,n$, and $\sum_{i=1}^{n} h_i=t$. Using the change of variables $s_i=h_i^{-1}\xi_i$, $i=1,\cdots,n$, we can rewrite (3) as

$$\int_0^\infty\cdots\int_0^\infty \exp(-\sum_{i=1}^{n} s_i)f(\sum_{i=1}^{n} h_is_i)ds_1\cdots ds_n \equiv J(h_1,\cdots,h_n). \tag{6}$$

Let m, n be positive integers with $m\leqq n$ and define

$$\Phi_n(t)=\{J(h_1,\cdots,h_n): \sum_{i=1}^{n} h_i=t,\ h_i\in(0,\infty),\ i=1,\cdots,n\},$$
$$\Phi_{n,m}(t)=\{J(h_1,\cdots,h_n): \sum_{i=1}^{n} h_i=t,\ h_i\in(0,t/m),\ i=1,\cdots,n\}.$$

Then the main results are summarized in the following form.

Theorem 1. *Let $m\leqq n$. Then we have:*

(i) $\min\Phi_n(t)=J(t/n,\cdots,t/n)$ *and* $\sup\Phi_{n,m}(t)=J(t/m,\cdots,t/m)$. *Therefore* $J(t/n,\cdots,t/n)\leqq J(h_1,\cdots,h_n)\leqq J(t/m,\cdots,t/m)$ *for* $h_i\in(0,t/m)$, $i=1,\cdots,n$ *such that* $\sum_{i=1}^{n} h_i=t$.

(ii) *The multiple integral* $J(t/m,\cdots,t/m)$ *can be written as the single integral* (4) *with* $h=t/m$. *Accordingly, if* $\int_0^\infty \xi^{m-1}e^{-\lambda\xi}f(\xi)d\xi<\infty$ *for* $\lambda>0$, *then* $(J(t/n,\cdots,t/n))_{n=m}^{\infty}$ *forms a strictly monotone decreasing sequence.*

Proof. First we observe that $J(h_1,\cdots,h_n)$ defines a (possibly extended real-valued) functional on the positive cone $(0,\infty)^n$ of $\boldsymbol{R}^n$. Since f is convex on $(0,\infty)$, we see that J is convex on $(0,\infty)^n$. Further, it follows from Fubini's theorem that $J(h_1,\cdots,h_n)$ is invariant under permutation of elements $h_1,\cdots,h_n$. Hence we have

$$J(h_1,h_2,\cdots,h_n)=J(h_2,\cdots,h_n,h_1)=\cdots=J(h_n,h_1,\cdots,h_{n-1}). \tag{7}$$

Let $h_i>0$, $i=1,\cdots,n$ and $\sum_{i=1}^{n} h_i=t$. Then, using (7) and the convexity of J on $(0,\infty)^n$, we obtain

$$\begin{aligned}J(h_1,\cdots,h_n)&=\frac{1}{n}J(h_1,\cdots,h_n)+\frac{1}{n}J(h_2,\cdots,h_n,h_1)+\cdots+\frac{1}{n}J(h_n,h_1,\cdots,h_{n-1})\\&\geqq J\Big(\frac{1}{n}(h_1,\cdots,h_n)+\frac{1}{n}(h_2,\cdots,h_n,h_1)+\cdots+\frac{1}{n}(h_n,h_1,\cdots,h_{n-1})\Big)\\&=J(t/n,\cdots,t/n).\end{aligned}$$

This proves the first assertion of (i). To prove the second assertion of

(i) we consider a polygon $P_{m,n}$ in $\boldsymbol{R}^n$ defined by

$$P_{m,n}=\{(h_1, \cdots, h_n): 0\leqq h_i\leqq t/m \text{ for } i=1, \cdots, n \text{ and } \textstyle\sum_{i=1}^n h_i=t\}.$$

The vertices of $P_{m,n}$ are n-dimensional vectors v such that m elements of v are equal to t/m and the other elements of v are 0. Hence there are ${}_nC_m$ vertices, say $v_1, \cdots, v_\nu$, $\nu={}_nC_m$. Let $0<h_i\leqq t/m$ and $\sum_{i=1}^n h_i=t$. Then $(h_1, \cdots, h_n)\in P_{m,n}$ and it is a convex combination of the vertices $v_1, \cdots, v_\nu$, namely

$$(h_1, \cdots, h_n)=\textstyle\sum_{k=1}^\nu \mu_k v_k, \quad \mu_k\geqq 0, \quad \sum_{k=1}^\nu \mu_k=1.$$

Further, a simple computation shows that $J(v_k)=J(t/m, \cdots, t/m)$ for $k=1, \cdots, \nu$. Hence we apply the convexity of J to get

$$J(h_1, \cdots, h_n)=J(\textstyle\sum_{k=1}^\nu \mu_k v_k)\leqq \sum_{k=1}^\nu \mu_k J(v_k)=J(t/m, \cdots, t/m).$$

From this the desired assertion follows. Assertion (i) states that if $J(t/m, \cdots, t/m)<\infty$ then the sequence $(J(t/n, \cdots, t/n))_{n=m}^\infty$ makes sense and is strictly monotone decreasing. Hence (ii) follows from Lemma 2 below.

q.e.d.

Lemma 2. *Let f be a nonnegative continuous function on* $(0, \infty)$. *Let $\lambda>0$, m a positive integer, and assume that* $\int_0^\infty \xi^{m-1}e^{-\lambda\xi}f(\xi)\,d\xi<\infty$. *Then we have*

$$\int_0^\infty\cdots\int_0^\infty \exp(-\lambda \textstyle\sum_{i=1}^m \xi_i)f(\sum_{i=1}^m \xi_i)\,d\xi_1\cdots d\xi_m \tag{8}$$
$$=(m-1)!^{-1}\int_0^\infty \xi^{m-1}e^{-\lambda\xi}f(\xi)\,d\xi.$$

Proof. We employ the change of variables $\eta_1=\xi_1$, $\eta_2=\xi_1+\xi_2$, $\cdots$, $\eta_m=\xi_1+\cdots+\xi_m$ to transform the left-hand side of (8) to

$$\int_{0<\eta_1<\eta_2<\cdots<\eta_m} \exp(-\lambda\eta_m)f(\eta_m)\,d\eta_1\cdots d\eta_m.$$

The application of Fubini's theorem now implies that this integral can be written as the iterated integral

$$\int_0^\infty \exp(-\lambda\eta_m)f(\eta_m)d\eta_m \int_0^{\eta_m}\cdots\int_0^{\eta_3}\int_0^{\eta_2} d\eta_1\cdots d\eta_{m-1}$$

which is nothing but the right-hand side of (8). q.e.d.

3. We here apply Theorem 1 to derive some characteristic properties of the infinitesimal generator of an analytic semigroup. Let A be the infinitesimal generator of a (C_0)-semigroup $\mathcal{T}$ on X such that $\|T(t)\|\leqq Me^{-\omega t}$ for $t\geqq 0$ and some $M\geqq 1$ and $\omega\geqq 0$.

Theorem 3. (a) *If $\mathcal{T}$ is analytic, then for every $\alpha>0$ there is a constant $C_\alpha>0$ such that* (5) *holds for $n\geqq m=[\alpha]+1$, $\theta=\alpha-[\alpha]$ and $h_1, \cdots, h_n$ with $0<h_j<m^{-1}\sum_{i=1}^n h_i$, $j=1, \cdots, n$.*

(b) *Conversely, suppose that there exists a sequence of partitions $\Delta_p=\{0=t_0^p<t_1^p<\cdots<t_{N(p)}^p=\tau_p\}$, $p=1, 2, \cdots$, satisfying*

$$\lim_{p\to\infty}\tau_p=\tau_0>0, \qquad \lim_{p\to\infty}\max_{1\leqq i\leqq N(p)}(t_i^p-t_{i-1}^p)=0,$$

and that (5) *holds for $\alpha=1$, $n=N(p)$, $h_i=t_i^p-t_{i-1}^p$, $i=1, \cdots, N(p)$, and $p=1, 2, \cdots$. Then $\mathcal{T}$ is an analytic semigroup.*

Proof. Since assertion (a) was already observed, it remains to prove (b). Let $t \in (0, \tau_0)$. Then there is a sequence $(t^p_{j(p)})$ converging to t and $\lim_{p\to\infty} \prod_{i=1}^{j(p)} (I-h_i^p A)^{-1}x = T(t)x$ for $x \in X$. If $x \in D(A)$, then

$$\|A \prod_{i=1}^{j(p)} (I-h_i^p A)^{-1}x\| = \|\prod_{i=1}^{j(p)} (I-h_i^p A)^{-1}Ax\| \leqq C_1(t^p_{j(p)})^{-1}\|x\|.$$

Therefore, using the fact that A is closed, we have $\|AT(t)x\| = \|T(t)Ax\| \leqq C_1 t^{-1}\|x\|$ for $x \in D(A)$. This shows (see [2, p. 62]) that $\mathcal{T}$ is analytic.

q.e.d.

References

[1] M. G. Crandall, A. Pazy, and L. Tartar: Remarks on generators of analytic semigroups. Israel J. Math., **32**, 363–374 (1979).

[2] A. Pazy: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, vol. 44, Springer-Verlag (1984).